GRADED STRUCTURES OF v-ADIC CARLITZ MULTIPLE POLYLOGARITHMS AND v-ADIC MULTIPLE ZETA VALUES

HUNG-CHUN TSUI

Abstract. In this paper, we study v-adic Carlitz multiple polylogarithms and v-adic multiple zeta values. We prove that, for every finite place v of $\mathbb{F}_q(\theta)$, the natural weight gives a graded structure on the algebras generated by these values. Moreover, we show that every $\overline{\mathbb{F}_q(\theta)}$-linear relation among these values arises from an $\mathbb{F}_q(\theta)$-linear relation. In particular, our results establish a function field analogue of the Furusho–Yamashita direct sum conjecture and extend Chen's work from finite places of degree one to arbitrary finite places.

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1. Introduction

1.1. **Classical background.** Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of integers. We let

$$\mathcal{I} := \{\varnothing\} \cup \bigcup_{r\geq 1} \mathbb{N}^r$$

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denote the set of indices and define

$$\mathrm{dep}(\mathfrak{s}) := r, \qquad \mathrm{wt}(\mathfrak{s}) := s_1 + \cdots + s_r, \qquad \mathrm{dep}(\varnothing) = \mathrm{wt}(\varnothing) = 0$$

for every $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. For an admissible non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, namely $s_1 \geq 2$, the classical *multiple zeta value* is defined by

$$\zeta(\mathfrak{s}) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

In depth one, these values recover the special values of the Riemann zeta function at positive integers. We refer the reader to [Zha16] for further background.

For $w \geq 1$, let $\mathcal{Z}_w$ denote the $\mathbb{Q}$-vector space spanned by all multiple zeta values of weight w, and let $\mathcal{Z}$ be the $\mathbb{Q}$-algebra generated by all multiple zeta values. Multiple zeta values satisfy both the stuffle and shuffle product relations [Zha16, §3]. In particular, we have $\mathcal{Z}_{w_1}\mathcal{Z}_{w_2} \subseteq \mathcal{Z}_{w_1+w_2}$ for any $w_1, w_2 \geq 1$.

Multiple zeta values can also be viewed as special values of one-variable multiple polylogarithms. Furusho [Fur04] constructed p-adic multiple polylogarithms by means of Coleman's p-adic iterated integration [Col82] and defined the corresponding *p-adic multiple zeta values* $\zeta_p(\mathfrak{s})$ through suitable limits at $z = 1$. Let $\mathcal{Z}_p$ be the $\mathbb{Q}$-algebra generated by all p-adic multiple zeta values, and for $w \geq 1$, let $\mathcal{Z}_{p,w}$ denote the $\mathbb{Q}$-vector space spanned by the p-adic multiple zeta values of weight w. The p-adic multiple zeta values also satisfy the stuffle and shuffle product relations [BF06]. In particular, we have $\mathcal{Z}_{p,w_1}\mathcal{Z}_{p,w_2} \subseteq \mathcal{Z}_{p,w_1+w_2}$ for any $w_1, w_2 \geq 1$.

Furusho and Yamashita formulated the following direct sum conjecture for p-adic multiple zeta values [Yam10, Conjecture 5].

Conjecture 1.1 (Furusho–Yamashita)**.** *For every prime number p,*

$$\mathcal{Z}_p = \mathbb{Q} \oplus \bigoplus_{w \geq 1} \mathcal{Z}_{p,w}$$

forms a graded $\mathbb{Q}$-algebra.

This conjecture is the p-adic analogue of the celebrated Goncharov's direct sum conjecture for classical multiple zeta values, which predicts that the weight gives a grading on the algebra of multiple zeta values (see, for example, [Gon97, Conjecture 4.2]).

1.2. **The function field counterpart.** Let p be a prime number, let q be a power of p, and put

$$A = \mathbb{F}_q[\theta], \qquad K = \mathbb{F}_q(\theta).$$

Let $K_\infty = \mathbb{F}_q((1/\theta))$ be the completion of K at the infinite place and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. We denote by $|\cdot|_\infty$ the ∞-adic absolute value on $\mathbb{C}_\infty$, normalized such that $|\theta|_\infty = q$. Fix an algebraic closure $\overline{K}$ of K together with an embedding $\overline{K} \hookrightarrow \mathbb{C}_\infty$.

For a non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, define

$$\mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},\infty} := \left\{ (u_1, \ldots, u_r) \in \mathbb{C}_\infty^r \;\middle|\; |u_i|_\infty < q^{\frac{s_i q}{q-1}} \text{ for } 1 \leq i \leq r \right\}.$$

For $\mathbf{u} = (u_1, \dots, u_r) \in \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},\infty}$, the *Carlitz multiple polylogarithm* associated with $\mathfrak{s}$ is defined in [Cha14] by

$$\mathrm{Li}_{\mathfrak{s}}(\mathbf{u}) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{u_1^{q^{i_1}} \cdots u_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in \mathbb{C}_\infty,$$

where $L_0 := 1$ and $L_i := \prod_{j=1}^{i}(\theta - \theta^{q^j})$ for all $i \geq 1$. We remark that these values satisfy the harmonic product relations, which preserve the total weight (see [Cha14, §6.2]).

On the other hand, Thakur [Tha04] introduced the *∞-adic multiple zeta values*

$$\zeta_A(\mathfrak{s}) := \sum_{\substack{a_1, \dots, a_r \in A \text{ monic} \\ \deg a_1 > \cdots > \deg a_r \geq 0}} \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty$$

for every non-empty index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$. Thakur [Tha10] proved that these values satisfy the q-shuffle product relations, which also preserve the total weight.

Chang [Cha14] proved that the algebra generated by Carlitz multiple polylogarithmic values at algebraic points is graded by weight. As a consequence, the algebra generated by ∞-adic multiple zeta values is also graded by weight. Moreover, he showed that every $\overline{K}$-linear relation among these values is generated by K-linear relations.

We now turn to the finite place setting. Let v be a finite place of K, let K_v be the completion of K at v, and let $\mathbb{C}_v$ be the completion of an algebraic closure of K_v. We denote by $|\cdot|_v$ the normalized v-adic absolute value on $\mathbb{C}_v$ and fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v$. For a non-empty index $\mathfrak{s} = (s_1, \dots, s_r)$, put

$$\mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v} := \{(u_1, \dots, u_r) \in \mathbb{C}_v^r \mid |u_1|_v < 1, \quad |u_i|_v \leq 1 \ \text{ for } 2 \leq i \leq r\}.$$

For $\mathbf{u} \in (\overline{K}^\times)^{\mathrm{dep}(\mathfrak{s})} \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$, the same defining series of $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})$ converges v-adically. We denote its value by $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v$. The harmonic product relations remain valid on this convergence domain by the same argument as in the ∞-adic setting (cf. [Cha14, §6.2]).

Chang and Mishiba [CM19] constructed v-adic analytic continuations of Carlitz multiple polylogarithms and Carlitz multiple star polylogarithms by means of logarithms of suitable t-modules. Using an ∞-adic presentation of multiple zeta values in terms of Carlitz multiple star polylogarithms at integral points, they then defined the corresponding *v-adic multiple zeta values* $\zeta_A(\mathfrak{s})_v$ for every index $\mathfrak{s} \in \mathcal{I}$ (see [CM21, §6.1]). Chang, Chen, and Mishiba [CCM22, Theorem 1.2.3] further proved that these values satisfy the same q-shuffle product relations as their ∞-adic counterparts.

For $w \geq 1$, let $\mathcal{L}_{v,w}$ and $\overline{\mathcal{L}}_{v,w}$ be respectively the K-span and the $\overline{K}$-span of all values

$$\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v, \qquad \mathrm{wt}(\mathfrak{s}) = w, \qquad \mathbf{u} \in (\overline{K}^\times)^{\mathrm{dep}(\mathfrak{s})} \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}.$$

Let $\mathcal{L}_v$ and $\overline{\mathcal{L}}_v$ be the K- and $\overline{K}$-algebras generated by these values, respectively. Likewise, let $\mathcal{Z}_{v,w}$ and $\overline{\mathcal{Z}}_{v,w}$ be respectively the K-span and the $\overline{K}$-span of the values $\zeta_A(\mathfrak{s})_v$ with $\mathrm{wt}(\mathfrak{s}) = w$, and let $\mathcal{Z}_v$ and $\overline{\mathcal{Z}}_v$ be the corresponding K- and $\overline{K}$-algebras.

When v is a finite place of degree one, Chen [Che25] established the v-adic analogue of Chang's linear independence results for Carlitz multiple polylogarithmic values at algebraic points and, as an application, the corresponding direct sum result for v-adic multiple zeta values. The purpose of the present paper is to extend these results to finite places of arbitrary degree.

1.3. **Main results.** Our main result establishes the function field analogue of the Furusho–Yamashita conjecture and, in particular, generalizes Chen's result [Che25] from degree one finite places to arbitrary finite places.

Theorem 1.2. *For every finite place v of K,*

$$\overline{\mathcal{L}}_v = \overline{K} \oplus \bigoplus_{w \geq 1} \overline{\mathcal{L}}_{v,w}$$

forms a graded $\overline{K}$-algebra. Moreover, the canonical map

$$\overline{K} \otimes_K \mathcal{L}_v \xrightarrow{\sim} \overline{\mathcal{L}}_v$$

is an isomorphism.

The corresponding result for v-adic multiple zeta values is as follows.

Theorem 1.3. *For every finite place v of K,*

$$\overline{\mathcal{Z}}_v = \overline{K} \oplus \bigoplus_{w \geq 1} \overline{\mathcal{Z}}_{v,w}$$

forms a graded $\overline{K}$-algebra. Moreover, the canonical map

$$\overline{K} \otimes_K \mathcal{Z}_v \xrightarrow{\sim} \overline{\mathcal{Z}}_v$$

is an isomorphism.

As an immediate consequence, we obtain the following transcendence statement.

Corollary 1.4. *Let v be a finite place of K. Then the following assertions hold.*

(i) For every non-empty index $\mathfrak{s} \in \mathcal{I}$ and every $\mathbf{u} \in (\overline{K}^\times)^{\mathrm{dep}(\mathfrak{s})} \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$, the value $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v$ is either zero or transcendental over K.

(ii) For every non-empty index $\mathfrak{s} \in \mathcal{I}$, the v-adic multiple zeta value $\zeta_A(\mathfrak{s})_v$ is either zero or transcendental over K.

The main idea and strategy of the proof originate in Chang's work [Cha14] and were subsequently adapted to the v-adic setting by Chen [Che25]. However, several new ingredients are required when $\deg v > 1$. In the degree one case, the relevant deformation series is a one-variable series and its specializations are scalar. For a finite place of arbitrary degree d, we introduce the auxiliary function field determined by $v(T) = t^{-1}$ and suitable polynomials $\Delta_v(T, a)$ and $\mathcal{D}_{v,i}(T)$ in order to construct a two-variable deformation of v-adic Carlitz multiple polylogarithms (see §2.2 and §2.3). Passing to trace coordinates then produces a finite-dimensional Frobenius system to which the v-adic Anderson–Brownawell–Papanikolas criterion can be applied (see Proposition 3.5). Also, the specializations at the points $t = v^{-q^N}$ are no longer scalar, so additional nondegeneracy arguments are also needed to recover the linear independence conclusions (see Lemmas 3.12 and A.1). When $d = 1$, these additional constructions reduce to those appearing in Chen's proof. Once these ingredients are established, the proofs of the main theorems follow the strategy of Chang and Chen and are given in Appendix A.

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2. The v-adic Setting

2.1. **v-adic Anderson–Thakur series and ABP criterion.** We also denote by v the monic irreducible polynomial in A associated with the finite place v. For every $n \in \mathbb{Z}$, we define the n-fold Frobenius twist

$$(-)^{(n)} : \mathbb{C}_v((t)) \longrightarrow \mathbb{C}_v((t))$$

by

$$\left(\sum_i a_i t^i\right)^{(n)} := \sum_i a_i^{q^n} t^i.$$

Then the map $(-)^{(n)}$ is an $\mathbb{F}_q[t]$-algebra automorphism of $\mathbb{C}_v((t))$.

Following [Che25, §2.1], the ring of v-adic entire series is defined by

$$\mathcal{E}_v := \left\{ \sum_{i=0}^{\infty} a_i t^i \in \overline{K}[\![t]\!] \;\middle|\; \lim_{i\to\infty} |a_i|_v^{1/i} = 0,\ [K_v(a_0, a_1, \ldots) : K_v] < \infty \right\}.$$

Thus every element of $\mathcal{E}_v$ defines an entire function on $\mathbb{C}_v$, and its coefficients are contained in a fixed finite extension of K_v.

We also recall the v-adic Anderson–Thakur series introduced in [Che25, Definition 3.1.1]. For $\alpha \in \overline{K}^\times$ satisfying $|\alpha|_v < 1$, put

$$\Omega_\alpha(t) := \prod_{i\geq 1} \left(1 - \alpha^{q^i} t\right).$$

By [Che25, Proposition 3.1.3], one has

$$\Omega_\alpha(t) \in \mathcal{E}_v, \qquad \Omega_\alpha^{(-1)}(t) = (1-\alpha t)\Omega_\alpha(t), \qquad \Omega_\alpha^{(n)}(t) = \frac{\Omega_\alpha(t)}{\prod_{j=1}^{n}(1-\alpha^{q^j}t)} \quad (n \geq 0). \tag{1}$$

Moreover, we let

$$\widetilde{\pi}_\alpha := \Omega_\alpha(\alpha^{-1}) \in \mathbb{C}_v^\times.$$

We will use the following theorem due to Chen [Che25], which is the v-adic analogue of the Anderson–Brownawell–Papanikolas criterion in the ∞-adic setting (see [ABP04]).

Theorem 2.1 ([Che25, Theorem 2.2.2])**.** *Let*

$$\Phi(t) \in \mathrm{Mat}_\ell(\overline{K}[t]), \qquad \psi(t) \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E}_v)$$

satisfy $\psi^{(-1)} = \Phi\psi$. Let $\gamma \in \overline{K}^\times \setminus \overline{\mathbb{F}}_q^\times$ and suppose that

$$\det\Phi\left(\gamma^{(-i)}\right) \neq 0 \qquad \textit{for every } i \in \mathbb{N}.$$

If $\rho \in \mathrm{Mat}_{1\times\ell}(\overline{K})$ satisfies

$$\rho\psi(\gamma) = 0,$$

then there exists $P(t) \in \mathrm{Mat}_{1\times\ell}(\overline{K}[t])$ such that

$$P(\gamma) = \rho \qquad \textit{and} \qquad P(t)\psi(t) = 0.$$

In particular, an $\overline{K}$-linear relation among the entries of $\psi(\gamma)$ can be lifted to a polynomial relation among the entries of $\psi(t)$.

2.2. **Auxiliary function field and polynomials.** Put $d := \deg v$ and consider the function field

$$H_v := \mathrm{Frac}\left(\overline{K}[t,T]/(tv(T)-1)\right).$$

Throughout what follows, the variables T and t satisfy

$$v(T) = t^{-1}. \tag{2}$$

Equivalently, we may identify

$$H_v = \overline{K}(T), \qquad t = v(T)^{-1},$$

and regard H_v as a separable extension of $\overline{K}(t)$. Note that

$$[H_v : \overline{K}(t)] = [\overline{K}(T) : \overline{K}(v(T))] = d,$$

and $\{1, T, \dots, T^{d-1}\}$ is an $\overline{K}(t)$-basis of H_v.

We also consider the scalar extension

$$\mathbb{H}_v := H_v \otimes_{\overline{K}(t)} \mathbb{C}_v((t)) \cong \mathbb{C}_v((t))[T]/(v(T)-t^{-1}).$$

Thus $\mathbb{H}_v$ is a d-dimensional $\mathbb{C}_v((t))$-algebra with basis $\{1, T, \dots, T^{d-1}\}$.

For every $n \in \mathbb{Z}$, we extend the n-fold Frobenius twist to H_v and $\mathbb{H}_v$ by fixing T. Since $v(T) \in \mathbb{F}_q[T]$, the relation (2) is preserved by Frobenius twisting, and hence we obtain automorphisms

$$(-)^{(n)} : H_v \longrightarrow H_v, \qquad (-)^{(n)} : \mathbb{H}_v \longrightarrow \mathbb{H}_v.$$

Explicitly, if

$$F = \sum_{j=0}^{d-1} f_j(t) T^j \in \mathbb{H}_v,$$

then

$$F^{(n)} = \sum_{j=0}^{d-1} f_j^{(n)}(t) T^j.$$

For $a \in \overline{K}$, define

$$\Delta_v(T,a) := \frac{v(T)-v(a)}{T-a} \in \overline{K}[T].$$

Since $v(T) \in \mathbb{F}_q[T]$, for every $n \in \mathbb{Z}$ we have

$$\Delta_v(T,a)^{(n)} = \Delta_v(T, a^{q^n}). \tag{3}$$

We also put

$$\delta_v(T) := \Delta_v(T,\theta).$$

For every $i \geq 0$, define

$$\mathcal{D}_{v,0}(T) := 1, \qquad \mathcal{D}_{v,i}(T) := \prod_{j=1}^{i} \Delta_v(T, \theta^{q^j}) \quad (i \geq 1). \tag{4}$$

The following identities will be used repeatedly.

Lemma 2.2. *For every $i \geq 1$, we have*

$$\mathcal{D}_{v,i}^{(-1)}(T) = \delta_v(T)\mathcal{D}_{v,i-1}(T). \tag{5}$$

Moreover,

$$\mathcal{D}_{v,i}(\theta) = \prod_{j=1}^{i} \frac{v - v^{q^j}}{\theta - \theta^{q^j}}. \tag{6}$$

Proof. By (3), we have

$$\mathcal{D}_{v,i}^{(-1)}(T) = \prod_{j=1}^{i} \Delta_v(T, \theta^{q^{j-1}}) = \Delta_v(T,\theta) \prod_{j=1}^{i-1} \Delta_v(T, \theta^{q^j}) = \delta_v(T)\mathcal{D}_{v,i-1}(T),$$

which proves (5). On the other hand, since $v \in \mathbb{F}_q[\theta]$,

$$v(\theta^{q^j}) = v(\theta)^{q^j} = v^{q^j}.$$

Therefore, we obtain

$$\Delta_v(\theta, \theta^{q^j}) = \frac{v - v^{q^j}}{\theta - \theta^{q^j}},$$

and (6) follows immediately. □

Remark 2.3. If $\deg v = 1$, then

$$v(T) - v(a) = T - a,$$

and hence

$$\Delta_v(T, a) = 1, \qquad \delta_v(T) = 1, \qquad \mathcal{D}_{v,i}(T) = 1$$

for every $i \geq 0$. Thus these auxiliary polynomials disappear in the degree one case.

2.3. **The series $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)$.** Taking $\alpha = v$ in the notation of §2.1, we write $\Omega_v(t)$ and

$$\widetilde{\pi}_v := \Omega_v(v^{-1}).$$

Definition 2.4. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$ be non-empty and let $\mathbf{u} = (u_1, \dots, u_r) \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$. For $0 \leq \ell \leq r$, put

$$w_\ell := s_1 + \cdots + s_\ell, \qquad w_0 := 0, \qquad w := w_r = \mathrm{wt}(\mathfrak{s}).$$

We define

$$\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t) := \sum_{i_1 > \cdots > i_r \geq 0} \prod_{\ell=1}^{r} \left((\Omega_v^{s_\ell} u_\ell)^{(i_\ell)} \mathcal{D}_{v,i_\ell}(T)^{s_\ell} t^{i_\ell s_\ell} \right). \tag{7}$$

For the empty index, we set

$$\mathfrak{L}^{[v]}_{\varnothing,\varnothing}(T,t) := 1.$$

We first verify that the series in (7) indeed defines an element of $\mathbb{H}_v$.

Lemma 2.5. *For every non-empty $\mathfrak{s} \in \mathcal{I}$ and $\mathbf{u} \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$, the series $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)$ converges in $\mathbb{H}_v$ with respect to the t-adic topology.*

Proof. Consider a summand of (7) indexed by $i_1 > \cdots > i_r \geq 0$, and put $n := i_1 s_1 + \cdots + i_r s_r$. Writing

$$v(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \qquad (a_i \in \mathbb{F}_q),$$

the relation $v(T) = t^{-1}$ gives

$$T^d = t^{-1} - a_{d-1}T^{d-1} - \cdots - a_0.$$

Thus, a power T^m can be reduced to the basis $\{1, T, \ldots, T^{d-1}\}$ using coefficients whose pole order at $t = 0$ is at most $\lfloor m/d \rfloor$. Since $\deg_T \mathcal{D}_{v,i} = i(d-1)$, we have

$$\deg_T \prod_{\ell=1}^{r} \mathcal{D}_{v,i_\ell}(T)^{s_\ell}$$

is at most $(d-1)n$. Hence, after reducing this product to the basis $\{1, T, \ldots, T^{d-1}\}$, each of its coefficients has pole order at most

$$\left\lfloor \frac{(d-1)n}{d} \right\rfloor.$$

On the other hand, the same summand contains the factor t^n, while $(\Omega_v^{s_\ell} u_\ell)^{(i_\ell)} \in \mathbb{C}_v[\![t]\!]$. It follows that every basis coefficient of this summand has t-adic order at least

$$n - \left\lfloor \frac{(d-1)n}{d} \right\rfloor = \left\lceil \frac{n}{d} \right\rceil.$$

Note that $n \to \infty$ as $i_1 \to \infty$ and the t-adic orders of all basis coefficients tend to infinity, so the summands tend to zero in $\mathbb{H}_v$. Therefore $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)$ converges in $\mathbb{H}_v$. □

The factors $\mathcal{D}_{v,i}(T)$ are chosen so that (7) satisfies the following Frobenius difference equation.

Proposition 2.6. *Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ be non-empty and $\mathbf{u} = (u_1, \ldots, u_r) \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$. Put $\mathfrak{s}' = (s_1, \ldots, s_{r-1})$ and $\mathbf{u}' = (u_1, \ldots, u_{r-1})$. Then*

$$\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)^{(-1)} = u_r^{q^{-1}}(1-vt)^{s_r}\Omega_v^{s_r}\,(t\delta_v(T))^{w_{r-1}}\,\mathfrak{L}^{[v]}_{\mathfrak{s}',\mathbf{u}'}(T,t) + (t\delta_v(T))^{w}\,\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t).$$

Proof. Apply $(-)^{(-1)}$ to (7) and separate the summation into the cases $i_r = 0$ and $i_r \geq 1$.

If $i_r \geq 1$, then $i_\ell \geq 1$ for every ℓ. By Lemma 2.2, $\mathcal{D}^{(-1)}_{v,i_\ell}(T) = \delta_v(T)\mathcal{D}_{v,i_\ell-1}(T)$. After replacing each i_ℓ by $i_\ell - 1$, we obtain

$$(t\delta_v(T))^{w}\,\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t).$$

If $i_r = 0$, then $i_\ell \geq 1$ for every $\ell < r$. Shifting i_ℓ to $i_\ell - 1$ for $\ell < r$ gives $(t\delta_v(T))^{w_{r-1}}\,\mathfrak{L}^{[v]}_{\mathfrak{s}',\mathbf{u}'}(T,t)$, while

$$(\Omega_v^{s_r}u_r)^{(-1)} = u_r^{q^{-1}}(1-vt)^{s_r}\Omega_v^{s_r}$$

by (1). Combining the two cases proves the result. □

Specializing at $T = \theta$ recovers the corresponding v-adic Carlitz multiple polylogarithm.

Proposition 2.7. *Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ be non-empty and $\mathbf{u} = (u_1, \ldots, u_r) \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$. We have*

$$\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\theta, v^{-1}) = \widetilde{\pi}_v^{\,w}\,\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v. \tag{8}$$

Proof. By (1), for every $i \geq 0$,

$$\Omega_v^{(i)}(t) = \frac{\Omega_v(t)}{\prod_{j=1}^{i}(1 - v^{q^j}t)},$$

and hence

$$\Omega_v^{(i)}(v^{-1})v^{-i} = \frac{\widetilde{\pi}_v}{\prod_{j=1}^{i}(v - v^{q^j})}.$$

By Lemma 2.2,

$$\mathcal{D}_{v,i}(\theta) = \prod_{j=1}^{i} \frac{v - v^{q^j}}{\theta - \theta^{q^j}}.$$

Therefore, we obtain

$$\Omega_v^{(i)}(v^{-1})v^{-i}\mathcal{D}_{v,i}(\theta) = \frac{\widetilde{\pi}_v}{L_i}.$$

Applying this identity to each factor in (7) gives (8). □

Remark 2.8. If $\deg v = 1$, then $\delta_v(T) = 1$ and $\mathcal{D}_{v,i}(T) = 1$ for every $i \geq 0$. Hence $\mathfrak{L}_{\mathfrak{s},\mathbf{u}}^{[v]}(T,t)$ is independent of T and coincides with Chen's series in [Che25, Definition 3.2.1].

3. Frobenius Systems

3.1. **Trace coordinates.** Inside H_v, we have

$$t\delta_v(T) = t\frac{v(T) - v}{T - \theta} = \frac{1 - vt}{T - \theta}. \tag{9}$$

Let $M_v(t)$ be the matrix representing multiplication by $t\delta_v(T)$ with respect to the ordered basis $\{1, T, \dots, T^{d-1}\}$. That is,

$$t\delta_v(T)T^j = \sum_{i=0}^{d-1} (M_v(t))_{ij} T^i \qquad (0 \leq j \leq d-1). \tag{10}$$

Lemma 3.1. *We have*

$$M_v(t) \in \mathrm{Mat}_d(K[t])$$

and

$$\det M_v(t) = (-1)^{d+1} t(1 - vt)^{d-1}. \tag{11}$$

Proof. Write

$$v(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \qquad (a_i \in \mathbb{F}_q).$$

By (2),

$$tT^d = 1 - t(a_{d-1}T^{d-1} + \cdots + a_0). \tag{12}$$

Since $\delta_v(T) \in K[T]$ has degree $d-1$, repeated use of (12) expresses $t\delta_v(T)T^j$ in the basis $\{1, T, \dots, T^{d-1}\}$ with coefficients in $K[t]$. Hence $M_v(t) \in \mathrm{Mat}_d(K[t])$.

For the determinant, multiplication by $t\delta_v(T)$ on H_v has determinant equal to its field norm. Let β run through the roots of $v(X) - t^{-1}$, which is the minimal polynomial of T over $\overline{K}(t)$. Then

$$\det M_v(t) = \mathrm{Nr}_{H_v/\overline{K}(t)}\left(t\delta_v(T)\right) = t^d \prod_{v(\beta)=t^{-1}} \frac{v(\beta) - v}{\beta - \theta} = t^d \frac{(t^{-1} - v)^d}{\prod_\beta (\beta - \theta)}.$$

Since

$$v(X) - t^{-1} = \prod_{\beta}(X - \beta),$$

evaluation at $X = \theta$ gives

$$\prod_{\beta}(\beta - \theta) = (-1)^d(v - t^{-1}) = (-1)^{d+1}(t^{-1} - v).$$

Therefore, we obtain

$$\det M_v(t) = (-1)^{d+1}t(1 - vt)^{d-1}.$$

□

Recall that

$$\mathbb{H}_v = H_v \otimes_{\overline{K}(t)} \mathbb{C}_v((t)) \cong \mathbb{C}_v((t))[T]/(v(T) - t^{-1}).$$

We note that the field trace

$$\mathrm{Tr}_{H_v/\overline{K}(t)} : H_v \longrightarrow \overline{K}(t)$$

extends uniquely to a $\mathbb{C}_v((t))$-linear map

$$\mathrm{Tr} : \mathbb{H}_v \longrightarrow \mathbb{C}_v((t)).$$

For $F \in \mathbb{H}_v$, define

$$\mathbf{T}(F) := \begin{pmatrix} \mathrm{Tr}(F) \\ \mathrm{Tr}(TF) \\ \vdots \\ \mathrm{Tr}(T^{d-1}F) \end{pmatrix} \in \mathrm{Mat}_{d\times 1}\left(\mathbb{C}_v((t))\right). \tag{13}$$

The trace coordinates are compatible with multiplication by $t\delta_v(T)$ and with Frobenius twisting.

Lemma 3.2. *For every $m \geq 0$ and $F \in \mathbb{H}_v$, we have*

$$\mathbf{T}\left((t\delta_v(T))^m F\right) = \left(M_v(t)^{\mathsf{T}}\right)^m \mathbf{T}(F). \tag{14}$$

Moreover,

$$\mathbf{T}(F)^{(-1)} = \mathbf{T}\left(F^{(-1)}\right). \tag{15}$$

Proof. For $0 \leq j \leq d-1$, (10) gives

$$\mathrm{Tr}\left(T^j t\delta_v(T)F\right) = \mathrm{Tr}\left((t\delta_v(T)T^j)F\right) = \sum_{i=0}^{d-1}(M_v(t))_{ij}\,\mathrm{Tr}(T^iF).$$

This is the j-th entry of $M_v(t)^{\mathsf{T}}\mathbf{T}(F)$, and iteration gives (14).

For the second assertion, the Frobenius twist preserves both H_v and $\overline{K}(t)$. Since the field trace is compatible with field automorphisms, we have

$$\mathrm{Tr}\left(F^{(-1)}\right) = \mathrm{Tr}(F)^{(-1)}$$

for every $F \in H_v$, and hence also for every $F \in \mathbb{H}_v$. □

We next show that the trace coordinates of $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)$ are v-adic entire.

Lemma 3.3. *For $m \geq 0$, put $R_m(t) := \mathrm{Tr}(T^m)$. Then*

$$R_m(t) \in \mathbb{F}_q[t^{-1}],$$

and the pole order of $R_m(t)$ at $t = 0$ is at most $\lfloor m/d \rfloor$. In particular,

$$R_m(t) \in \mathbb{F}_q \qquad (0 \leq m < d).$$

Proof. For $1 \leq m < d$, Newton's identities applied to

$$v(X) - t^{-1} = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + (a_0 - t^{-1})$$

give

$$R_m + a_{d-1}R_{m-1} + \cdots + a_{d-m+1}R_1 + m a_{d-m} = 0.$$

Since $d - m \geq 1$, the constant coefficient $a_0 - t^{-1}$ does not occur. Hence $R_m \in \mathbb{F}_q$. Moreover, $R_0 = \mathrm{Tr}(1) = d \in \mathbb{F}_q$. For $m \geq d$, the relation $v(T) = t^{-1}$ gives

$$T^m = -a_{d-1}T^{m-1} - \cdots - a_0 T^{m-d} + t^{-1}T^{m-d}.$$

Taking traces yields

$$R_m = -a_{d-1}R_{m-1} - \cdots - a_0 R_{m-d} + t^{-1}R_{m-d}.$$

The assertion follows by induction on m. □

Proposition 3.4. *Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ be non-empty and $\mathbf{u} = (u_1, \ldots, u_r) \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$. For every $0 \leq m \leq d-1$, write*

$$T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t) = \sum_{j=0}^{d-1} f_{m,j}(t) T^j.$$

Then

$$f_{m,j}(t) \in \mathcal{E}_v \qquad (0 \leq j \leq d-1), \qquad \text{and} \qquad \mathrm{Tr}\left(T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right) \in \mathcal{E}_v.$$

Consequently,

$$\mathbf{T}\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right) \in \mathrm{Mat}_{d\times 1}(\mathcal{E}_v).$$

Proof. Fix $0 \leq m \leq d-1$. Consider a summand indexed by $i_1 > \cdots > i_r \geq 0$, and put $n := i_1 s_1 + \cdots + i_r s_r$. By the proof of Lemma 2.5 and Lemma 3.3, the pole order at $t = 0$ of each basis coefficient of

$$T^m \prod_{\ell=1}^{r} \mathcal{D}_{v,i_\ell}(T)^{s_\ell}$$

is at most

$$\left\lfloor \frac{m + (d-1)n}{d} \right\rfloor.$$

Also, since

$$\prod_{\ell=1}^{r} (\Omega_v^{s_\ell} u_\ell)^{(i_\ell)} \in \mathbb{C}_v[\![t]\!],$$

the corresponding summand of each $f_{m,j}(t)$ has t-adic order at least

$$n - \left\lfloor \frac{m + (d-1)n}{d} \right\rfloor \geq 0.$$

This lower bound tends to infinity as $n \to \infty$. In particular, no negative powers of t occur.

We next prove that each $f_{m,j}(t)$ is entire. Fix $R > 0$. By the definition of Ω_v, for every $i \geq 0$ we have

$$\Omega_v^{(i)}(t) = \prod_{k>i}(1 - v^{q^k}t).$$

Hence, for $|t|_v \leq R$,

$$|\Omega_v^{(i)}(t)|_v \leq \prod_{k\geq 1} \max\{1, |v|_v^{q^k} R\},$$

uniformly in i. The product on the right is finite, since $|v|_v^{q^k} R < 1$ for all sufficiently large k.

On the other hand, since

$$\Delta_v(T, \theta^{q^k}) \in A[T],$$

all coefficients of

$$\prod_{\ell=1}^{r} \mathcal{D}_{v,i_\ell}(T)^{s_\ell}$$

have v-adic absolute value at most 1. By the proof of Lemma 2.5, each basis coefficient of

$$T^m \prod_{\ell=1}^{r} \mathcal{D}_{v,i_\ell}(T)^{s_\ell}$$

belongs to $A[t^{-1}]$ and has all coefficients of v-adic absolute value at most 1. Hence, after multiplication by t^n, each such basis coefficient is a polynomial in t of degree at most n. Consequently, on $|t|_v \leq R$, its absolute value is at most $\max\{1, R\}^n$.

Now, since $\mathbf{u} \in \mathbb{D}_{\mathbf{s},v}^{\mathrm{Conv}}$, we have

$$|u_1|_v < 1, \qquad |u_\ell|_v \leq 1 \quad (2 \leq \ell \leq r).$$

Thus, on $|t|_v \leq R$, the absolute value of the corresponding summand of each $f_{m,j}(t)$ is at most

$$\left(\prod_{k\geq 1} \max\{1, |v|_v^{q^k} R\}\right)^w |u_1|_v^{q^{i_1}} \max\{1, R\}^n.$$

Since $n \leq w i_1$, this is bounded above by

$$\left(\prod_{k\geq 1} \max\{1, |v|_v^{q^k} R\}\right)^w |u_1|_v^{q^{i_1}} \max\{1, R\}^{w i_1},$$

which tends to 0 as $i_1 \to \infty$. For each fixed i_1, there are only finitely many tuples $i_1 > \cdots > i_r \geq 0$. Hence, grouping the summands according to i_1, the defining series of each $f_{m,j}(t)$ converges uniformly on every closed disk $|t|_v \leq R$. Since $R > 0$ was arbitrary, each $f_{m,j}(t)$ defines an entire function on $\mathbb{C}_v$.

It remains to verify the algebraicity for the coefficients. By (1), $\Omega_v \in \mathcal{E}_v$ and its coefficients lie in a finite extension of K_v. Note that for every $i \geq 0$, the coefficients of $\Omega_v^{(i)}$ lie in this extension. Enlarging this extension if necessary, we may also assume that it contains $u_1, \ldots, u_r$. Furthermore, the coefficients of $\mathcal{D}_{v,i}(T)$ belong to K. Hence the coefficients of every summand of each $f_{m,j}(t)$ lie in the same finite extension of K_v. Finally, the lower bound

$$n - \left\lfloor \frac{m + (d-1)n}{d} \right\rfloor \longrightarrow \infty \qquad (n \to \infty)$$

shows that, for each fixed $N \geq 0$, only finitely many tuples $i_1 > \cdots > i_r \geq 0$ can contribute to the coefficient of t^N. Therefore,

$$f_{m,j}(t) \in \mathcal{E}_v \qquad (0 \leq j \leq d-1).$$

Finally, since

$$T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t) = \sum_{j=0}^{d-1} f_{m,j}(t) T^j,$$

we have

$$\mathrm{Tr}\left(T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right) = \sum_{j=0}^{d-1} f_{m,j}(t) R_j(t).$$

By Lemma 3.3, $R_j(t) \in \mathbb{F}_q$ for all $0 \leq j \leq d-1$. Hence, we obtain

$$\mathrm{Tr}\left(T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right) \in \mathcal{E}_v.$$

The final assertion follows from the definition of $\mathbf{T}$. □

3.2. **Frobenius difference equations.** Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ be non-empty and $\mathbf{u} = (u_1, \ldots, u_r) \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$. We put $w = \mathrm{wt}(\mathfrak{s})$. For $1 \leq \ell \leq r$, set

$$\mathfrak{s}_\ell := (s_1, \ldots, s_\ell), \qquad \mathbf{u}_\ell := (u_1, \ldots, u_\ell), \qquad w_\ell := s_1 + \cdots + s_\ell.$$

Also put $w_0 := 0$ and

$$\boldsymbol{\tau}_v := \mathbf{T}(1) = \begin{pmatrix} \mathrm{Tr}(1) \\ \mathrm{Tr}(T) \\ \vdots \\ \mathrm{Tr}(T^{d-1}) \end{pmatrix} \in \mathrm{Mat}_{d\times 1}(\mathbb{F}_q).$$

Note that the entries of $\boldsymbol{\tau}_v$ lie in $\mathbb{F}_q$ by Lemma 3.3.

Define

$$\psi_{\mathfrak{s},\mathbf{u}}(t) := \begin{pmatrix} \Omega_v^w \\ \mathbf{T}\left(\Omega_v^{w-w_1} \mathfrak{L}^{[v]}_{\mathfrak{s}_1,\mathbf{u}_1}(T,t)\right) \\ \vdots \\ \mathbf{T}\left(\Omega_v^{w-w_r} \mathfrak{L}^{[v]}_{\mathfrak{s}_r,\mathbf{u}_r}(T,t)\right) \end{pmatrix} = \begin{pmatrix} \Omega_v^w \\ \Omega_v^{w-w_1} \mathbf{T}(\mathfrak{L}^{[v]}_{\mathfrak{s}_1,\mathbf{u}_1}(T,t)) \\ \vdots \\ \Omega_v^{w-w_r} \mathbf{T}(\mathfrak{L}^{[v]}_{\mathfrak{s}_r,\mathbf{u}_r}(T,t)) \end{pmatrix}. \tag{16}$$

Note that $w = w_r$, so the last d-entry of (16) is $\mathbf{T}(\mathfrak{L}^{[v]}_{\mathfrak{s}_r,\mathbf{u}_r}(T,t))$. Also, by (1) and Proposition 3.4, we have

$$\psi_{\mathfrak{s},\mathbf{u}}(t) \in \mathrm{Mat}_{(1+rd)\times 1}(\mathcal{E}_v).$$

Proposition 3.5. *Let the notation be as above. We have*

$$\psi^{(-1)}_{\mathfrak{s},\mathbf{u}} = \Phi_{\mathfrak{s},\mathbf{u}} \psi_{\mathfrak{s},\mathbf{u}}, \tag{17}$$

where $\Phi_{\mathfrak{s},\mathbf{u}} \in \mathrm{Mat}_{1+rd}(\overline{K}[t])$ *is the block lower triangular matrix*

$$\Phi_{\mathfrak{s},\mathbf{u}}(t) = \left(\begin{array}{c|c|c|c|c} (1-vt)^w & 0 & 0 & \cdots & 0 \\ \hline u_1^{q^{-1}}(1-vt)^w \boldsymbol{\tau}_v & \begin{array}{c}(1-vt)^{w-w_1}\\ (M_v(t)^{\mathsf{T}})^{w_1}\end{array} & 0 & \cdots & 0 \\ \hline 0 & \begin{array}{c}u_2^{q^{-1}}(1-vt)^{w-w_1}\\ (M_v(t)^{\mathsf{T}})^{w_1}\end{array} & \begin{array}{c}(1-vt)^{w-w_2}\\ (M_v(t)^{\mathsf{T}})^{w_2}\end{array} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & \begin{array}{c}u_r^{q^{-1}}(1-vt)^{w-w_{r-1}}\\ (M_v(t)^{\mathsf{T}})^{w_{r-1}}\end{array} & (M_v(t)^{\mathsf{T}})^w \end{array}\right).$$

Here, the first diagonal block is of size 1×1*, the first subdiagonal block is of size* $d\times 1$*, and all the remaining displayed blocks are of size* $d\times d$*.*

Proof. By (1), the first coordinate satisfies

$$(\Omega_v^w)^{(-1)} = (1-vt)^w \Omega_v^w.$$

For $1 \le \ell \le r$, applying Proposition 2.6 to $\mathfrak{L}^{[v]}_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(T,t)$ gives

$$\begin{aligned}&\left(\Omega_v^{w-w_\ell}\mathfrak{L}^{[v]}_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(T,t)\right)^{(-1)}\\ &\quad= u_\ell^{q^{-1}}(1-vt)^{w-w_{\ell-1}}\left(t\delta_v(T)\right)^{w_{\ell-1}}\Omega_v^{w-w_{\ell-1}}\mathfrak{L}^{[v]}_{\mathfrak{s}_{\ell-1},\mathbf{u}_{\ell-1}}(T,t)\\ &\qquad+(1-vt)^{w-w_\ell}\left(t\delta_v(T)\right)^{w_\ell}\Omega_v^{w-w_\ell}\mathfrak{L}^{[v]}_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(T,t),\end{aligned}$$

where, when $\ell = 1$, the first term on the right-hand side is understood as

$$u_1^{q^{-1}}(1-vt)^w\Omega_v^w.$$

Applying $\mathbf{T}$ and using Lemma 3.2, we obtain, for $2 \le \ell \le r$,

$$\begin{aligned}&\mathbf{T}\left(\Omega_v^{w-w_\ell}\mathfrak{L}^{[v]}_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(T,t)\right)^{(-1)}\\ &\quad= u_\ell^{q^{-1}}(1-vt)^{w-w_{\ell-1}}\left(M_v(t)^{\mathsf{T}}\right)^{w_{\ell-1}}\mathbf{T}\left(\Omega_v^{w-w_{\ell-1}}\mathfrak{L}^{[v]}_{\mathfrak{s}_{\ell-1},\mathbf{u}_{\ell-1}}(T,t)\right)\\ &\qquad+(1-vt)^{w-w_\ell}\left(M_v(t)^{\mathsf{T}}\right)^{w_\ell}\mathbf{T}\left(\Omega_v^{w-w_\ell}\mathfrak{L}^{[v]}_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(T,t)\right).\end{aligned}$$

For $\ell = 1$, since $\mathbf{T}(\Omega_v^w) = \boldsymbol{\tau}_v\Omega_v^w$, we obtain

$$\begin{aligned}&\mathbf{T}\left(\Omega_v^{w-w_1}\mathfrak{L}^{[v]}_{\mathfrak{s}_1,\mathbf{u}_1}(T,t)\right)^{(-1)}\\ &\quad= u_1^{q^{-1}}(1-vt)^w\boldsymbol{\tau}_v\Omega_v^w + (1-vt)^{w-w_1}\left(M_v(t)^{\mathsf{T}}\right)^{w_1}\mathbf{T}\left(\Omega_v^{w-w_1}\mathfrak{L}^{[v]}_{\mathfrak{s}_1,\mathbf{u}_1}(T,t)\right).\end{aligned}$$

These identities give precisely (17). □

Corollary 3.6. *Let the notation be as above. The pair* $(\Phi_{\mathfrak{s},\mathbf{u}}, \psi_{\mathfrak{s},\mathbf{u}})$ *satisfies the hypotheses of the v-adic ABP criterion in Theorem 2.1 at* $\gamma = v^{-1}$*.*

Proof. It is clear that $v^{-1} \in \overline{K}^\times \setminus \overline{\mathbb{F}}_q^\times$. By construction, notice that $\psi_{\mathfrak{s},\mathbf{u}} \in \mathrm{Mat}_{(1+rd)\times 1}(\mathcal{E}_v)$ and $\Phi_{\mathfrak{s},\mathbf{u}} \in \mathrm{Mat}_{1+rd}(\overline{K}[t])$. Since $\Phi_{\mathfrak{s},\mathbf{u}}$ is block lower triangular, its determinant is the product of the determinants of its diagonal blocks. By Lemma 3.1,

$$\det M_v(t) = (-1)^{d+1}t(1-vt)^{d-1},$$

so every zero of $\det \Phi_{\mathfrak{s},\mathbf{u}}(t)$ is contained in $\{0, v^{-1}\}$. For every $i \geq 1$, the element $(v^{-1})^{(-i)}$ is neither 0 nor v^{-1}. Therefore, we have

$$\det \Phi_{\mathfrak{s},\mathbf{u}} \left((v^{-1})^{(-i)} \right) \neq 0 \qquad (i \geq 1),$$

and the assertion follows from Theorem 2.1. □

3.3. **Specialization.** In this subsection, we study the specialization of $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)$ and its trace coordinates at $t = v^{-q^N}$ for $N \geq 1$.

We first need the following lemmas.

Lemma 3.7. *Let $a \in \overline{K}^\times$ and suppose that $v(T) - a^{-1}$ is separable. Let*

$$F(T,t) = \sum_{j=0}^{d-1} f_j(t) T^j \in \mathbb{H}_v$$

be such that each $f_j(t) \in \mathbb{C}_v((t))$ is defined at $t = a$. Then

$$\mathrm{Tr}(F(T,t))(a) = \sum_{v(\beta)=a^{-1}} F(\beta, a), \tag{18}$$

where the sum is taken over all roots of $v(T) - a^{-1}$.

Proof. Let $L(t)$ denote the matrix representing multiplication by T with respect to the basis $\{1, T, \ldots, T^{d-1}\}$. Then

$$R_j(t) = \mathrm{Tr}(T^j) = \mathrm{tr}\left(L(t)^j \right).$$

After specializing $t = a$, the matrix $L(a)$ represents multiplication by T on

$$\overline{K}[T]/(v(T) - a^{-1}).$$

Since $v(T) - a^{-1}$ is separable, $L(a)$ is diagonalizable over $\overline{K}$, with eigenvalues given by the roots β of $v(T) - a^{-1}$. Hence,

$$R_j(a) = \mathrm{tr}\left(L(a)^j \right) = \sum_{v(\beta)=a^{-1}} \beta^j.$$

Therefore,

$$\mathrm{Tr}(F(T,t))(a) = \sum_{j=0}^{d-1} f_j(a) R_j(a) = \sum_{j=0}^{d-1} f_j(a) \sum_{v(\beta)=a^{-1}} \beta^j = \sum_{v(\beta)=a^{-1}} F(\beta, a).$$

□

We next compare the factors occurring in $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)$ at $t = v^{-q^N}$ with those at $t = v^{-1}$.

Lemma 3.8. *For $N \geq 1$ and $h \geq 0$, we have*

$$\Omega_v^{(N+h)}(t) t^{N+h} \Big|_{t=v^{-q^N}} = v^{-Nq^N} \left(\Omega_v^{(h)}(t) t^h \Big|_{t=v^{-1}} \right)^{q^N}. \tag{19}$$

Moreover, if $i < N$, then

$$\Omega_v^{(i)}(v^{-q^N}) = 0. \tag{20}$$

Proof. Using

$$\Omega_v^{(i)}(t) = \prod_{j>i}(1 - v^{q^j}t),$$

if $i < N$, the factor corresponding to $j = N$ vanishes at $t = v^{-q^N}$, which proves (20). For $i = N + h$, we have

$$\begin{aligned}\Omega_v^{(N+h)}(v^{-q^N})v^{-(N+h)q^N} &= v^{-(N+h)q^N}\prod_{j>N+h}(1 - v^{q^j-q^N}) \\ &= v^{-Nq^N}\left(v^{-h}\prod_{j>h}(1 - v^{q^j-1})\right)^{q^N},\end{aligned}$$

which gives (19). □

For $N \geq 1$, put

$$\lambda_{v,N} := v^{-Nq^N}\prod_{j=1}^{N}\Delta_v(\theta^{q^N}, \theta^{q^j}) \in \overline{K}^\times. \tag{21}$$

Here, note that for $j < N$,

$$v(\theta^{q^N}) = v^{q^N} \neq v^{q^j} = v(\theta^{q^j}),$$

and

$$\Delta_v(\theta^{q^N}, \theta^{q^N}) = v'(\theta^{q^N}) \neq 0.$$

Lemma 3.9. *For $N \geq 1$ and $h \geq 0$, we have*

$$\mathcal{D}_{v,N+h}(\theta^{q^N}) = v^{Nq^N}\lambda_{v,N}\mathcal{D}_{v,h}(\theta)^{q^N}. \tag{22}$$

If $\beta \neq \theta^{q^N}$ is another root of $v(T) - v^{q^N}$, then

$$\mathcal{D}_{v,i}(\beta) = 0 \qquad (i \geq N). \tag{23}$$

Proof. For (22), notice that

$$\mathcal{D}_{v,N+h}(\theta^{q^N}) = \left(\prod_{j=1}^{N}\Delta_v(\theta^{q^N}, \theta^{q^j})\right)\prod_{m=1}^{h}\Delta_v(\theta^{q^N}, \theta^{q^{N+m}}) = v^{Nq^N}\lambda_{v,N}\mathcal{D}_{v,h}(\theta)^{q^N}.$$

Here, we use the fact that $\Delta_v(\theta^{q^N}, \theta^{q^{N+m}}) = \Delta_v(\theta, \theta^{q^m})^{q^N}$.

Now, let $\beta \neq \theta^{q^N}$ be another root of $v(T) - v^{q^N}$. The factor corresponding to $j = N$ is

$$\Delta_v(\beta, \theta^{q^N}) = \frac{v(\beta) - v(\theta^{q^N})}{\beta - \theta^{q^N}} = 0.$$

Hence, we obtain (23), which completes the proof. □

Combining the preceding three lemmas, we determine the specialization of $\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T, t)$ on the fiber $v(T) = v^{q^N}$.

Proposition 3.10. *Let $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ be non-empty and let $\mathbf{u} = (u_1, \ldots, u_r) \in (\overline{K}^\times)^r \cap \mathbb{D}^{\mathrm{Conv}}_{\mathfrak{s},v}$. Let $w = \mathrm{wt}(\mathfrak{s})$ and $N \geq 1$. For every root β of $v(T) - v^{q^N}$, we have*

$$\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\beta, v^{-q^N}) = \begin{cases}\lambda^w_{v,N}\left(\widetilde{\pi}^w_v\,\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v\right)^{q^N}, & \beta = \theta^{q^N}, \\ 0, & \beta \neq \theta^{q^N}.\end{cases} \tag{24}$$

Consequently,

$$\mathbf{T}\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)(v^{-q^N}) = \lambda^w_{v,N}\left(\widetilde{\pi}^w_v \operatorname{Li}_{\mathfrak{s}}(\mathbf{u})_v\right)^{q^N}\mathbf{V}_N, \tag{25}$$

where

$$\mathbf{V}_N := \begin{pmatrix} 1 \\ \theta^{q^N} \\ \vdots \\ (\theta^{q^N})^{d-1} \end{pmatrix}. \tag{26}$$

Proof. By Lemma 3.8, every summand for which some $i_\ell < N$ vanishes at $t = v^{-q^N}$. Thus only indices of the form

$$i_\ell = N + h_\ell, \qquad h_1 > \cdots > h_r \geq 0,$$

survive.

Suppose first that $\beta = \theta^{q^N}$. Applying Lemmas 3.8 and 3.9 to each factor gives

$$\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\theta^{q^N}, v^{-q^N}) = \lambda^w_{v,N}\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\theta, v^{-1})\right)^{q^N}.$$

By Proposition 2.7, we obtain

$$\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\theta^{q^N}, v^{-q^N}) = \lambda^w_{v,N}\left(\widetilde{\pi}^w_v \operatorname{Li}_{\mathfrak{s}}(\mathbf{u})_v\right)^{q^N}.$$

Now let $\beta \neq \theta^{q^N}$ be another root of $v(T) - v^{q^N}$. By Lemma 3.9,

$$\mathcal{D}_{v,i}(\beta) = 0 \qquad (i \geq N).$$

Hence every remaining summand vanishes at $(T,t) = (\beta, v^{-q^N})$, and therefore

$$\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\beta, v^{-q^N}) = 0.$$

This proves (24).

We next note that $v(T) - v^{q^N}$ is separable. Since v is irreducible over the perfect field $\mathbb{F}_q$, we have $v'(T) \neq 0$. If β were a common root of $v(T) - v^{q^N}$ and $v'(T)$, then β would be algebraic over $\mathbb{F}_q$. Hence $v(\beta)$ would also be algebraic over $\mathbb{F}_q$, contradicting $v(\beta) = v^{q^N}$. Therefore, by Proposition 3.4 and Lemma 3.7 for $0 \leq m \leq d-1$, we obtain

$$\begin{aligned} &\operatorname{Tr}\left(T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)(v^{-q^N}) \\ &\quad = \sum_{v(\beta)=v^{q^N}} \beta^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\beta, v^{-q^N}) \\ &\quad = (\theta^{q^N})^m \lambda^w_{v,N}\left(\widetilde{\pi}^w_v \operatorname{Li}_{\mathfrak{s}}(\mathbf{u})_v\right)^{q^N}, \end{aligned}$$

which proves (25), as desired. □

We next recover the value at $t = v^{-1}$ from the trace coordinates. Write

$$\frac{\Delta_v(T,\theta)}{v'(\theta)} = \sum_{m=0}^{d-1} \eta_{v,m} T^m, \qquad \boldsymbol{\eta}_v := (\eta_{v,0}, \ldots, \eta_{v,d-1}). \tag{27}$$

Lemma 3.11. *We have*

$$\boldsymbol{\eta}_v \mathbf{T}\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)(v^{-1}) = \widetilde{\pi}^w_v \operatorname{Li}_{\mathfrak{s}}(\mathbf{u})_v. \tag{28}$$

Proof. As in the proof of Proposition 3.10, $v(T) - v$ is separable. Hence, by Proposition 3.4 and Lemma 3.7,

$$\begin{aligned}\boldsymbol{\eta}_v \mathbf{T}\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)(v^{-1}) &= \sum_{m=0}^{d-1} \eta_{v,m} \operatorname{Tr}\left(T^m \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)(v^{-1}) \\ &= \sum_{v(\beta)=v} \left(\sum_{m=0}^{d-1} \eta_{v,m}\beta^m\right) \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\beta, v^{-1}) \\ &= \sum_{v(\beta)=v} \frac{\Delta_v(\beta,\theta)}{v'(\theta)} \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\beta, v^{-1}).\end{aligned}$$

Since

$$\frac{\Delta_v(\theta,\theta)}{v'(\theta)} = 1, \qquad \Delta_v(\beta,\theta) = 0$$

for every other root β of $v(T) - v$, we obtain

$$\boldsymbol{\eta}_v \mathbf{T}\left(\mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(T,t)\right)(v^{-1}) = \mathfrak{L}^{[v]}_{\mathfrak{s},\mathbf{u}}(\theta, v^{-1}).$$

The result now follows from Proposition 2.7. □

Finally, we end this section with the following lemma.

Lemma 3.12. *Let*

$$Q(t) = (Q_0(t), \dots, Q_{d-1}(t)) \in \operatorname{Mat}_{1\times d}(\overline{K}[t])$$

be nonzero. Then

$$Q(v^{-q^N})\mathbf{V}_N \neq 0$$

for all but finitely many $N \geq 1$.

Proof. Suppose that

$$Q(v^{-q^N})\mathbf{V}_N = 0$$

for infinitely many N. Consider the rational function

$$R(T) := \sum_{m=0}^{d-1} Q_m(v(T)^{-1})T^m \in \overline{K}(T).$$

For every such N, we have

$$R(\theta^{q^N}) = \sum_{m=0}^{d-1} Q_m(v^{-q^N})(\theta^{q^N})^m = Q(v^{-q^N})\mathbf{V}_N = 0.$$

Since the points θ^{q^N} are pairwise distinct, we obtain $R(T) = 0$. By the linear independence of $\{1, T, \dots, T^{d-1}\}$ over $\overline{K}(t)$, we see that for every $0 \leq m \leq d-1$,

$$Q_m(t) = Q_m(v(T)^{-1}) = 0,$$

which is a contradiction. □

Appendix A. Proof of the Main Theorems

The proof follows the strategy of [Cha14, §4] and its v-adic adaptation in [Che25, §3.1]. The main difference in the present setting is that, for a finite place of arbitrary degree, the special values are recovered from the trace coordinates constructed in the previous section. We study linear relations among v-adic Carlitz multiple polylogarithmic values and then deduce the corresponding results for v-adic multiple zeta values.

A.1. **Proof of Theorem 1.2.** We first treat linear relations among values of the same weight.

Lemma A.1. *Let $S(T) \in \overline{K}(T)$ and $c \in \overline{K}$. Suppose that $S(\theta^{q^N}) = c^{q^N}$ for all sufficiently large N for which the left-hand side is defined. Then $S(T) \in \mathbb{F}_q(T)$. Moreover, if S is regular at $T = \theta$, then $S(\theta) \in K$.*

Proof. For all sufficiently large N, we have

$$S^{(1)}(\theta^{q^{N+1}}) = S(\theta^{q^N})^q = c^{q^{N+1}} = S(\theta^{q^{N+1}}).$$

Hence, the rational function $S^{(1)} - S$ has infinitely many zeros, and therefore $S^{(1)} = S$.

Write $S = B/C$, where $B, C \in \overline{K}[T]$ are relatively prime and C is monic. Then

$$B^{(1)}C = BC^{(1)}.$$

It follows that $C \mid C^{(1)}$. Since C and $C^{(1)}$ are monic of the same degree, we have $C^{(1)} = C$, and consequently $B^{(1)} = B$. Thus $B, C \in \mathbb{F}_q[T]$, so $S \in \mathbb{F}_q(T)$. If S is regular at $T = \theta$, then $S(\theta) \in \mathbb{F}_q(\theta) = K$. □

Lemma A.2. *Let $Z_1, \dots, Z_m$ be nonzero convergent v-adic Carlitz multiple polylogarithmic values at algebraic points of the same weight. If $Z_1, \dots, Z_m$ are linearly independent over K, then they are linearly independent over $\overline{K}$.*

Proof. Suppose on the contrary that $Z_1, \dots, Z_m$ are linearly dependent over $\overline{K}$. Replacing them by a minimally dependent subset and relabeling if necessary, we may assume that

$$Z_1 + c_2 Z_2 + \cdots + c_m Z_m = 0, \qquad c_2, \dots, c_m \in \overline{K}, \tag{29}$$

and that $Z_2, \dots, Z_m$ are linearly independent over $\overline{K}$. Put $c_1 := 1$, and let w denote their common weight.

For each $1 \le j \le m$, let

$$\psi_j^{(-1)} = \Phi_j \psi_j$$

be the Frobenius system constructed in Proposition 3.5 for Z_j. We form the block system

$$\psi := \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_m \end{pmatrix}, \qquad \Phi := \operatorname{diag}(\Phi_1, \dots, \Phi_m).$$

Then

$$\psi^{(-1)} = \Phi \psi.$$

By Corollary 3.6, the pair (Φ, ψ) satisfies the hypotheses of the v-adic Anderson–Brownawell–Papanikolas criterion at $t = v^{-1}$.

For each $1 \leq j \leq m$, let r_j denote the depth of the index corresponding to Z_j, so that

$$\psi_j \in \mathrm{Mat}_{(1+r_jd)\times 1}(\mathcal{E}_v).$$

Define

$$\rho_j := (0, \ldots, 0, c_j\boldsymbol{\eta}_v) \in \mathrm{Mat}_{1\times(1+r_jd)}(\overline{K}),$$

where the initial zero block has length $1 + (r_j - 1)d$. We then set

$$\rho := (\rho_1 \mid \cdots \mid \rho_m).$$

By Lemma 3.11, we have

$$\rho\psi(v^{-1}) = \widetilde{\pi}_v^w \sum_{j=1}^{m} c_j Z_j = 0.$$

Hence, by Theorem 2.1, there exists a polynomial row $\mathbf{p}(t)$ such that

$$\mathbf{p}(v^{-1}) = \rho, \qquad \mathbf{p}(t)\psi(t) = 0. \tag{30}$$

Write

$$\mathbf{p}(t) = (\mathbf{p}_1(t) \mid \cdots \mid \mathbf{p}_m(t)),$$

where $\mathbf{p}_j(t)$ is the block corresponding to ψ_j. For each j, let

$$Q_j(t) \in \mathrm{Mat}_{1\times d}(\overline{K}[t])$$

denote the last d entries of $\mathbf{p}_j(t)$. Thus, $Q_j(t)$ is the block of $\mathbf{p}(t)$ paired with the last trace coordinate block of ψ_j. Then

$$Q_j(v^{-1}) = c_j\boldsymbol{\eta}_v. \tag{31}$$

In particular, $Q_1(v^{-1}) = \boldsymbol{\eta}_v \neq 0$, and hence $Q_1(t) \neq 0$.

For $N \geq 1$, put

$$a_{j,N} := Q_j(v^{-q^N})\mathbf{V}_N.$$

By Lemma 3.12, $a_{1,N} \neq 0$ for all sufficiently large N. By (20) with $i = 0$, we have

$$\Omega_v(v^{-q^N}) = 0,$$

it follows that the first coordinate and all the trace coordinate blocks of $\psi_j(v^{-q^N})$ except the last one vanish. Moreover, by Proposition 3.10, we have

$$\mathbf{T}(\mathfrak{L}^{[v]}_{\mathfrak{s}_j,\mathbf{u}_j}(T,t))(v^{-q^N}) = \lambda^w_{v,N}\left(\widetilde{\pi}^w_v Z_j\right)^{q^N}\mathbf{V}_N.$$

Hence,

$$\psi_j(v^{-q^N}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda^w_{v,N}\left(\widetilde{\pi}^w_v Z_j\right)^{q^N}\mathbf{V}_N \end{pmatrix}.$$

Specializing the second identity in (30) at $t = v^{-q^N}$, we obtain

$$\lambda^w_{v,N}(\widetilde{\pi}^w_v)^{q^N}\sum_{j=1}^{m} a_{j,N}Z_j^{q^N} = 0,$$

and hence

$$\sum_{j=1}^{m} a_{j,N}^{q^{-N}} Z_j = 0.$$

Since $Z_2, \dots, Z_m$ are linearly independent over $\overline{K}$, the space of $\overline{K}$-linear relations among $Z_1, \dots, Z_m$ is one-dimensional. Comparing this relation with (29), we obtain

$$\frac{a_{j,N}}{a_{1,N}} = c_j^{q^N} \qquad (1 \le j \le m) \tag{32}$$

for all sufficiently large N.

For $1 \le j \le m$, define

$$A_j(T) := Q_j(v(T)^{-1}) \begin{pmatrix} 1 \\ T \\ \vdots \\ T^{d-1} \end{pmatrix} \in \overline{K}(T).$$

Since $Q_1(t) = Q_1(v(T)^{-1}) \neq 0$ and $1, T, \dots, T^{d-1}$ are linearly independent over $\overline{K}(t)$, we have $A_1(T) \neq 0$. Moreover,

$$A_j(\theta^{q^N}) = Q_j(v^{-q^N})\mathbf{V}_N = a_{j,N}.$$

Therefore, the rational function

$$S_j(T) := \frac{A_j(T)}{A_1(T)}$$

satisfies

$$S_j(\theta^{q^N}) = c_j^{q^N}$$

for all sufficiently large N. By Lemma A.1, we have

$$S_j(T) \in \mathbb{F}_q(T).$$

Finally, by (27) and (31),

$$A_j(\theta) = c_j \boldsymbol{\eta}_v \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{d-1} \end{pmatrix} = c_j \frac{\Delta_v(\theta, \theta)}{v'(\theta)} = c_j.$$

In particular, $A_1(\theta) = 1$. Hence, S_j is regular at $T = \theta$, and Lemma A.1 gives $c_j = S_j(\theta) \in K$. Thus (29) is a nontrivial K-linear relation among $Z_1, \dots, Z_m$, which is a contradiction. □

We next separate values of distinct weights.

Lemma A.3. *Let $w_1 > \cdots > w_s$ be distinct positive integers. For each $1 \le i \le s$, let $V_i = \{Z_{i1}, \dots, Z_{im_i}\}$ be a finite set of nonzero convergent v-adic Carlitz multiple polylogarithmic values at algebraic points of weight w_i. If each V_i is linearly independent over K, then*

$$\{1\} \cup \bigcup_{i=1}^{s} V_i$$

is linearly independent over $\overline{K}$.

Proof. By Lemma A.2, each V_i is linearly independent over $\overline{K}$. Suppose on the contrary that

$$a_0 + \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} Z_{ij} = 0, \qquad a_0, a_{ij} \in \overline{K}, \tag{33}$$

is a nontrivial relation. After discarding the weights which do not occur, we may assume that $a_{1j} \neq 0$ for some j.

For each Z_{ij}, let $\psi_{ij}^{(-1)} = \Phi_{ij}\psi_{ij}$ be the Frobenius system given by Proposition 3.5. We homogenize these systems to weight w_1 by setting

$$\widehat{\psi}_{ij} := \Omega_v^{w_1 - w_i} \psi_{ij}, \qquad \widehat{\Phi}_{ij} := (1 - vt)^{w_1 - w_i} \Phi_{ij}. \tag{34}$$

Then $\widehat{\psi}_{ij}^{(-1)} = \widehat{\Phi}_{ij}\widehat{\psi}_{ij}$. We form the block system

$$\widehat{\psi} := \begin{pmatrix} \widehat{\psi}_{11} \\ \vdots \\ \widehat{\psi}_{1m_1} \\ \widehat{\psi}_{21} \\ \vdots \\ \widehat{\psi}_{sm_s} \end{pmatrix}, \qquad \widehat{\Phi} := \operatorname{diag}\left(\widehat{\Phi}_{11}, \dots, \widehat{\Phi}_{1m_1}, \widehat{\Phi}_{21}, \dots, \widehat{\Phi}_{sm_s}\right).$$

Then

$$\widehat{\psi}^{(-1)} = \widehat{\Phi}\widehat{\psi}.$$

As in Corollary 3.6, this system satisfies the hypotheses of the v-adic Anderson–Brownawell–Papanikolas criterion at $t = v^{-1}$.

For each (i, j), let r_{ij} denote the depth of the index corresponding to Z_{ij}. Define

$$\rho_{11} := (a_0, 0, \dots, 0, a_{11}\boldsymbol{\eta}_v) \in \operatorname{Mat}_{1 \times (1 + r_{11}d)}(\overline{K}),$$

where the middle zero block has length $(r_{11} - 1)d$, and, for $(i, j) \neq (1, 1)$, define

$$\rho_{ij} := \left(0, 0, \dots, 0, a_{ij}\boldsymbol{\eta}_v\right) \in \operatorname{Mat}_{1 \times (1 + r_{ij}d)}(\overline{K}),$$

where again the middle zero block has length $(r_{ij} - 1)d$. According to the ordering of the blocks in $\widehat{\psi}$, set

$$\rho := \left(\rho_{11} \mid \cdots \mid \rho_{1m_1} \mid \rho_{21} \mid \cdots \mid \rho_{sm_s}\right).$$

Since $\Omega_v(v^{-1}) = \widetilde{\pi}_v$, we obtain from Lemma 3.11 that

$$\rho\widehat{\psi}(v^{-1}) = \widetilde{\pi}_v^{w_1}\left(a_0 + \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij} Z_{ij}\right) = 0.$$

By Theorem 2.1, there exists a polynomial row $\mathbf{p}(t)$ such that

$$\mathbf{p}(v^{-1}) = \rho, \qquad \mathbf{p}(t)\widehat{\psi}(t) = 0. \tag{35}$$

For each $1 \leq j \leq m_1$, let $Q_j(t) \in \operatorname{Mat}_{1 \times d}(\overline{K}[t])$ be the subrow of $\mathbf{p}(t)$ corresponding to the last d entries of the system associated with Z_{1j}. Choose j_0 such that $a_{1j_0} \neq 0$. Then

$$Q_{j_0}(v^{-1}) = a_{1j_0}\boldsymbol{\eta}_v \neq 0,$$

so $Q_{j_0}(t) \neq 0$. By Lemma 3.12,

$$Q_{j_0}(v^{-q^N})\mathbf{V}_N \neq 0$$

for all sufficiently large N.

Fix such an N and specialize the second identity in (35) at $t = v^{-q^N}$. If $i > 1$, by (20), $\Omega_v^{w_1-w_i}(v^{-q^N}) = 0$, so every coordinate of $\widehat{\psi}_{ij}(v^{-q^N})$ vanishes. For $i = 1$, all coordinates except the final trace coordinates also vanish. Hence Proposition 3.10 gives

$$0 = \lambda_{v,N}^{w_1} (\widetilde{\pi}_v^{w_1})^{q^N} \sum_{j=1}^{m_1} \left(Q_j(v^{-q^N}) \mathbf{V}_N \right) Z_{1j}^{q^N}.$$

The scalar outside the sum is nonzero, and the coefficient corresponding to j_0 is nonzero. It follows that $Z_{11}^{q^N}, \dots, Z_{1m_1}^{q^N}$ are linearly dependent over $\overline{K}$. Taking q^N-th roots gives a nontrivial $\overline{K}$-linear relation among $Z_{11}, \dots, Z_{1m_1}$, which is a contradiction. □

We can now prove the first main theorem.

Proof of Theorem 1.2. We first prove that the weight decomposition is direct. Suppose that

$$a_0 + G_{w_1} + \cdots + G_{w_s} = 0,$$

where $w_1, \dots, w_s$ are distinct positive integers, $a_0 \in \overline{K}$, and $G_{w_i} \in \overline{\mathcal{L}}_{v,w_i}$. For each nonzero G_{w_i}, choose finitely many weight w_i v-adic Carlitz multiple polylogarithmic values at algebraic points whose $\overline{K}$-span contains G_{w_i}, and replace them by a maximal K-linearly independent subset V_i. The K-span of V_i contains the original finite set, so the $\overline{K}$-span of V_i still contains G_{w_i}.

Expanding the relation in terms of these sets gives an $\overline{K}$-linear relation among $\{1\} \cup \bigcup_i V_i$. By Lemma A.3, this relation is trivial. Therefore

$$\overline{\mathcal{L}}_v = \overline{K} \oplus \bigoplus_{w \geq 1} \overline{\mathcal{L}}_{v,w}.$$

Since the harmonic product preserves the total weight, this decomposition makes $\overline{\mathcal{L}}_v$ a graded $\overline{K}$-algebra.

It remains to prove the base change assertion. Fix $w \geq 1$ and choose a K-basis $\{Z_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{L}_{v,w}$ consisting of v-adic Carlitz multiple polylogarithmic values at algebraic points. Every finite subset of this basis is K-linearly independent and therefore, by Lemma A.2, $\overline{K}$-linearly independent. Hence $\{Z_\lambda\}_{\lambda \in \Lambda}$ is linearly independent over $\overline{K}$. It follows that the canonical map

$$\overline{K} \otimes_K \mathcal{L}_{v,w} \longrightarrow \overline{\mathcal{L}}_{v,w}$$

is injective, while surjectivity follows from the definition of $\overline{\mathcal{L}}_{v,w}$. Thus it is an isomorphism for every $w \geq 1$. Summing over all weights and using the direct sum decomposition, we obtain

$$\overline{K} \otimes_K \mathcal{L}_v \xrightarrow{\sim} \overline{\mathcal{L}}_v.$$

□

A.2. **Proofs of Theorem 1.3 and Corollary 1.4.** We now deduce the corresponding statements for v-adic multiple zeta values. By [CM21, Theorem 5.2.5 and Definition 6.1.1], every v-adic multiple zeta value of weight w is expressed as a K-linear combination of v-adic Carlitz multiple star polylogarithmic values of weight w at integral points. By [CCM22, Remark 5.2.1], these values belong to the K-span of the convergent v-adic Carlitz multiple star polylogarithmic values of weight w. Moreover, by [Che25, Proposition 3.3.5], the K-spans of

the convergent v-adic Carlitz multiple star polylogarithmic values and the convergent v-adic Carlitz multiple polylogarithmic values of weight w coincide. Hence,

$$\mathcal{Z}_{v,w} \subseteq \mathcal{L}_{v,w}, \qquad \overline{\mathcal{Z}}_{v,w} \subseteq \overline{\mathcal{L}}_{v,w}. \tag{36}$$

Proof of Theorem 1.3. By (36) and Theorem 1.2, the sum

$$\overline{K} + \sum_{w \geq 1} \overline{\mathcal{Z}}_{v,w}$$

is direct. Therefore,

$$\overline{\mathcal{Z}}_v = \overline{K} \oplus \bigoplus_{w \geq 1} \overline{\mathcal{Z}}_{v,w}.$$

Together with the q-shuffle product relations for v-adic multiple zeta values, this makes $\overline{\mathcal{Z}}_v$ a graded $\overline{K}$-algebra.

It remains to prove the base change assertion. Fix $w \geq 1$, and let $Z_1, \dots, Z_m \in \mathcal{Z}_{v,w}$ be linearly independent over K. By (36), these elements belong to $\mathcal{L}_{v,w}$. The isomorphism

$$\overline{K} \otimes_K \mathcal{L}_{v,w} \xrightarrow{\sim} \overline{\mathcal{L}}_{v,w}$$

from Theorem 1.2 implies that $Z_1, \dots, Z_m$ remain linearly independent over $\overline{K}$. Consequently, the canonical map

$$\overline{K} \otimes_K \mathcal{Z}_{v,w} \longrightarrow \overline{\mathcal{Z}}_{v,w}$$

is injective. It is surjective by definition and hence is an isomorphism. Summing over all weights and using the direct sum decomposition, we obtain

$$\overline{K} \otimes_K \mathcal{Z}_v \xrightarrow{\sim} \overline{\mathcal{Z}}_v.$$

□

The transcendence statement in Corollary 1.4 now follows immediately.

Proof of Corollary 1.4. Let $\mathfrak{s} \in \mathcal{I}$ be non-empty and let $\mathbf{u} \in (\overline{K}^\times)^{\mathrm{dep}(\mathfrak{s})} \cap \mathbb{D}_{\mathfrak{s},v}^{\mathrm{Conv}}$. Suppose that $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v$ is nonzero and algebraic over K. Then $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v \in \overline{K}$. On the other hand, $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v \in \overline{\mathcal{L}}_{v,\mathrm{wt}(\mathfrak{s})}$, and hence Theorem 1.2 gives

$$\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v \in \overline{K} \cap \overline{\mathcal{L}}_{v,\mathrm{wt}(\mathfrak{s})} = \{0\},$$

a contradiction. Thus $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u})_v$ is either zero or transcendental over K.

Similarly, $\zeta_A(\mathfrak{s})_v$ is either zero or transcendental over K. □

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(Hung-Chun Tsui) Department of Mathematics, National Tsing Hua University, No. 101, Sec. 2, Guangfu Rd., East Dist., Hsinchu City 300044, Taiwan (R.O.C.)

Email address: hctsui@gapp.nthu.edu.tw