MISHIBA'S CONJECTURE ON THE COACTION OF ∞-ADIC MULTIPLE ZETA VALUES

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Abstract. We study multiple zeta values in positive characteristic. We construct, using ∞-adic multiple zeta values, a coaction of the ∞-adic multiple zeta value algebra modulo $\zeta_A(q-1)$ on the ∞-adic multiple zeta value algebra itself, and prove that it agrees with Mishiba's coaction constructed via special values of Carlitz multiple polylogarithms. We also determine the coaction on the $\star$-inverse values and show that the antipode on the ∞-adic multiple zeta value algebra modulo $\zeta_A(q-1)$ is given by the $\star$-inverse operation. In particular, these results prove Mishiba's conjecture concerning the coaction and the antipode for ∞-adic multiple zeta values.

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1. Introduction

1.1. **Classical Background.** Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of integers. We let

$$\mathcal{I} = \{\varnothing\} \cup \bigcup_{r\geq 1} \mathbb{N}^r$$

denote the set of indices and define

$$\mathrm{dep}(\mathfrak{s}) = r, \qquad \mathrm{wt}(\mathfrak{s}) = s_1 + \cdots + s_r, \qquad \mathrm{dep}(\varnothing) = \mathrm{wt}(\varnothing) = 0$$

for every $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. For an admissible non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, namely $s_1 \geq 2$, the classical *multiple zeta value* is defined by

$$\zeta(\mathfrak{s}) = \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}.$$

Multiple zeta values are known to satisfy the stuffle and shuffle product relations. A further structure appears at the motivic level. Following [Bro12], let $\mathfrak{Z}^{\mathfrak{m}}$ denote the graded $\mathbb{Q}$-algebra of *motivic multiple zeta values*, and write $\zeta^{\mathfrak{m}}(\mathfrak{s}) \in \mathfrak{Z}^{\mathfrak{m}}$ for the motivic multiple zeta value associated with an admissible index $\mathfrak{s}$. By [Bro12], the quotient

$$\overline{\mathfrak{Z}}^{\mathfrak{m}} = \mathfrak{Z}^{\mathfrak{m}} / \zeta^{\mathfrak{m}}(2)\mathfrak{Z}^{\mathfrak{m}}$$

carries a natural structure of a graded Hopf algebra, and $\mathfrak{Z}^{\mathfrak{m}}$ is naturally a graded comodule over $\overline{\mathfrak{Z}}^{\mathfrak{m}}$ through the motivic coaction

$$\Delta^{\mathfrak{m}} : \mathfrak{Z}^{\mathfrak{m}} \longrightarrow \overline{\mathfrak{Z}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathfrak{Z}^{\mathfrak{m}}.$$

Let $\mathfrak{Z}$ denote the $\mathbb{Q}$-algebra generated by the classical multiple zeta values. There is a surjective period homomorphism

$$\mathrm{per} : \mathfrak{Z}^{\mathfrak{m}} \longrightarrow \mathfrak{Z}$$

satisfying

$$\mathrm{per}\,(\zeta^{\mathfrak{m}}(\mathfrak{s})) = \zeta(\mathfrak{s}).$$

It is conjectured that this period map is an isomorphism [Bro14, §3.2]. This naturally raises the question of whether the motivic coaction admits a counterpart directly at the level of multiple zeta values, in particular after passing to the quotient by $\zeta(2)$.

1.2. **∞-adic Multiple Zeta Values.** We now turn to the function field setting. Let p be a prime number and let q be a power of p. Put

$$A = \mathbb{F}_q[\theta], \qquad K = \mathbb{F}_q(\theta),$$

and let $K_\infty = \mathbb{F}_q((1/\theta))$ be the completion of K at the infinite place. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞, and view a fixed algebraic closure $\overline{K}$ of K as a subfield of $\mathbb{C}_\infty$.

Thakur introduced the function field analogue of multiple zeta values $\zeta_A(\mathfrak{s})$ in [Tha04]. Throughout this paper, we refer to these values as the *∞-adic multiple zeta values*. In [Cha14], Chang introduced the *Carlitz multiple polylogarithms*. We will mainly consider their special values $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$, where $\mathbf{1}$ denotes the tuple $(1, \ldots, 1)$ of the appropriate depth. Recently, Mishiba [Mis26] introduced the corresponding $\star$-inverse values $\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s})$ and $\mathrm{Li}_{\mathfrak{s}}^{\star-\mathrm{inv}}(\mathbf{1})$. (For the precise definitions, see §2.1.)

Put $L_1 = \theta - \theta^q$ and fix an $\mathbb{F}_p(L_1)$-subalgebra R of $\overline{K}$. We let $\mathfrak{Z}_{\infty,R}$ denote the R-algebra generated by the ∞-adic multiple zeta values $\zeta_A(\mathfrak{s})$, with $\mathfrak{s} \in \mathcal{I}$. For each $w \geq 0$, we also write

$$\mathfrak{Z}_{\infty,R,w} = \mathrm{span}_R \left\{ \zeta_A(\mathfrak{s}) : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w \right\}.$$

We further put

$$\overline{\mathfrak{Z}}_{\infty,R} = \mathfrak{Z}_{\infty,R} / \zeta_A(q-1)\mathfrak{Z}_{\infty,R},$$

and let

$$\pi_\infty : \mathfrak{Z}_{\infty,R} \longrightarrow \overline{\mathfrak{Z}}_{\infty,R}$$

denote the quotient map.

For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ and $0 \leq i \leq r$, we define

$$\mathfrak{s}[:i] = (s_1, \ldots, s_i), \qquad \mathfrak{s}[i+1:] = (s_{i+1}, \ldots, s_r).$$

Any truncation outside the indicated range is understood to be $\varnothing$.

In [Mis26], Mishiba constructed a coaction on $\mathfrak{Z}_{\infty,R}$. The resulting structure may be summarized as follows.

Theorem 1.1 ([Mis26, Theorem 1.3.4]). *The following assertions hold.*

(i) There exists an R-algebra homomorphism

$$\widetilde{\Delta}^{\mathrm{Li}}_\infty : \mathfrak{Z}_{\infty,R} \longrightarrow \mathfrak{Z}_{\infty,R} \otimes_R \overline{\mathfrak{Z}}_{\infty,R}$$

such that, for every $\mathfrak{s} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{s}) = r$,

$$\widetilde{\Delta}^{\mathrm{Li}}_\infty \left(\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}) \right) = \sum_{i=0}^{r} \mathrm{Li}_{\mathfrak{s}[:i]}(\mathbf{1}) \otimes \pi_\infty(\mathrm{Li}_{\mathfrak{s}[i+1:]}(\mathbf{1})).$$

(ii) The map $\widetilde{\Delta}^{\mathrm{Li}}_\infty$ is a coaction, and $\overline{\mathfrak{Z}}_{\infty,R}$ carries a connected graded Hopf algebra structure compatible with this coaction.

(iii) The antipode

$$S_\infty : \overline{\mathfrak{Z}}_{\infty,R} \longrightarrow \overline{\mathfrak{Z}}_{\infty,R}$$

is a non-trivial R-algebra involution satisfying

$$S_\infty \left(\pi_\infty(\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})) \right) = \pi_\infty \left(\mathrm{Li}^{\star-\mathrm{inv}}_{\mathfrak{s}}(\mathbf{1}) \right)$$

for every $\mathfrak{s} \in \mathcal{I}$.

Mishiba further conjectured that the same maps are compatible with the ∞-adic multiple zeta values.

Conjecture 1.2 ([Mis26, Conjecture 1.3.7]). *For every $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, the following assertions hold.*

(i) The coaction $\widetilde{\Delta}^{\mathrm{Li}}_\infty$ satisfies

$$\widetilde{\Delta}^{\mathrm{Li}}_\infty \left(\zeta_A(\mathfrak{s}) \right) = \sum_{i=0}^{r} \zeta_A(\mathfrak{s}[:i]) \otimes \pi_\infty \left(\zeta_A(\mathfrak{s}[i+1:]) \right).$$

(ii) The antipode S_∞ satisfies

$$S_\infty \left(\pi_\infty(\zeta_A(\mathfrak{s})) \right) = \pi_\infty \left(\zeta^{\star-\mathrm{inv}}_A(\mathfrak{s}) \right).$$

1.3. **Main Results.** We develop a formalism based on evaluation maps and deconcatenation which treats the ζ- and Li-realizations, together with their $\star$-inverse counterparts, in a uniform manner. Applying this formalism to ∞-adic multiple zeta values, we construct a coaction directly from the ζ-realization and show that it coincides with Mishiba's coaction $\widetilde{\Delta}_\infty^{\mathrm{Li}}$. The same formalism also determines the coaction on the corresponding $\star$-inverse values and the action of the antipode.

Our main results are the following.

Theorem 1.3 (Restated as Theorem 4.2)**.** *Let R be an $\mathbb{F}_p(L_1)$-subalgebra of $\overline{K}$. There exists a well-defined R-algebra homomorphism*

$$\widetilde{\Delta}_\infty : \mathfrak{Z}_{\infty,R} \longrightarrow \mathfrak{Z}_{\infty,R} \otimes_R \overline{\mathfrak{Z}}_{\infty,R}$$

such that, for every $\mathfrak{s} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{s}) = r$,

$$\widetilde{\Delta}_\infty(\zeta_A(\mathfrak{s})) = \sum_{i=0}^{r} \zeta_A(\mathfrak{s}[:i]) \otimes \pi_\infty(\zeta_A(\mathfrak{s}[i+1:])).$$

Moreover, we have

$$\widetilde{\Delta}_\infty(\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})) = \sum_{i=0}^{r} \mathrm{Li}_{\mathfrak{s}[:i]}(\mathbf{1}) \otimes \pi_\infty(\mathrm{Li}_{\mathfrak{s}[i+1:]}(\mathbf{1})),$$

$$\widetilde{\Delta}_\infty(\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s})) = \sum_{i=0}^{r} \zeta_A^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]) \otimes \pi_\infty(\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s}[:i])),$$

and

$$\widetilde{\Delta}_\infty(\mathrm{Li}_{\mathfrak{s}}^{\star-\mathrm{inv}}(\mathbf{1})) = \sum_{i=0}^{r} \mathrm{Li}_{\mathfrak{s}[i+1:]}^{\star-\mathrm{inv}}(\mathbf{1}) \otimes \pi_\infty(\mathrm{Li}_{\mathfrak{s}[:i]}^{\star-\mathrm{inv}}(\mathbf{1})).$$

In particular, we have

$$\widetilde{\Delta}_\infty = \widetilde{\Delta}_\infty^{\mathrm{Li}}$$

and Conjecture 1.2 (i) holds.

Thus, after passing to the quotient by $\zeta_A(q-1)$, the usual deconcatenation of indices is compatible with the q-shuffle product and gives the coproduct on $\overline{\mathfrak{Z}}_{\infty,R}$.

The preceding theorem also determines the action of the antipode on the ζ-realization.

Theorem 1.4 (Restated as Theorem 4.4)**.** *The antipode*

$$S_\infty : \overline{\mathfrak{Z}}_{\infty,R} \longrightarrow \overline{\mathfrak{Z}}_{\infty,R}$$

satisfies, for every $\mathfrak{s} \in \mathcal{I}$,

$$S_\infty(\pi_\infty(\zeta_A(\mathfrak{s}))) = \pi_\infty(\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s})), \qquad S_\infty(\pi_\infty(\mathrm{Li}_{\mathfrak{s}}(\mathbf{1}))) = \pi_\infty(\mathrm{Li}_{\mathfrak{s}}^{\star-\mathrm{inv}}(\mathbf{1})).$$

In particular, Conjecture 1.2 (ii) holds.

Consequently, modulo $\zeta_A(q-1)$, the $\star$-inverse values satisfy exactly the same algebraic relations as the ordinary ∞-adic multiple zeta values.

The rest of the paper is organized as follows. In §2, we recall the necessary background on the ζ- and Li-realizations, their product structures, and the linear relations among the corresponding values. In §3, we first compare the ordinary and $\star$-inverse evaluations. We then study the behavior of the relations $\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})$ under deconcatenation and use these results to construct a formal coaction. Finally, in §4, we apply this formalism to ∞-adic

multiple zeta values, identify the resulting coaction with Mishiba's coaction arising from the Li-realization, determine the coaction on the $\star$-inverse values, and show that the antipode on $\overline{\mathfrak{Z}}_{\infty,R}$ is given by the $\star$-inverse operation.

2. Preliminaries

2.1. **Algebraic Setup.** Recall that

$$\mathcal{I} = \{\varnothing\} \cup \bigcup_{r\geq 1} \mathbb{N}^r$$

denotes the set of indices. We let

$$\mathfrak{H}_R = \mathrm{span}_R\{[\mathfrak{s}] : \mathfrak{s} \in \mathcal{I}\}$$

be the free R-module generated by the set of indices $\mathcal{I}$ and

$$\mathfrak{H}_{R,w} = \mathrm{span}_R\{[\mathfrak{s}] : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w\}$$

for all $w \geq 0$. We put $1 = [\varnothing]$. Then the concatenation map

$$[-,-] : \mathfrak{H}_R \times \mathfrak{H}_R \to \mathfrak{H}_R$$

is the unique R-bilinear map determined by

$$[\mathfrak{s},\mathfrak{n}] = [s_1,\dots,s_r,n_1,\dots,n_\ell], \qquad [\varnothing,\mathfrak{s}] = [\mathfrak{s},\varnothing] = [\mathfrak{s}], \qquad [\varnothing,\varnothing] = [\varnothing],$$

where $\mathfrak{s} = (s_1,\dots,s_r), \mathfrak{n} = (n_1,\dots,n_\ell) \in \mathcal{I}$ are non-empty indices. Iterated concatenations are understood R-multilinearly and are denoted by

$$[-,\dots,-]\colon \mathfrak{H}_R \times \cdots \times \mathfrak{H}_R \longrightarrow \mathfrak{H}_R.$$

We identify an index with its corresponding basis element when no confusion can arise.

On the other hand, the deconcatenation map is the unique R-linear map

$$\Delta : \mathfrak{H}_R \longrightarrow \mathfrak{H}_R \otimes_R \mathfrak{H}_R$$

such that

$$\Delta([\mathfrak{s}]) = \sum_{i=0}^{r} [\mathfrak{s}[:i]] \otimes [\mathfrak{s}[i+1:]]$$

for all indices $\mathfrak{s} \in \mathcal{I}$.

Following [CCM23, Mis26], we define the realization maps as follows. For $d \geq 0$ and $s \in \mathbb{N}$, set

$$S_d(s) = \sum_{\substack{a\in A \text{ monic} \\ \deg a = d}} \frac{1}{a^s}, \qquad L_0 = 1, \qquad L_d = \prod_{i=1}^{d}(\theta - \theta^{q^i}) \quad (d \geq 1).$$

For $\bullet \in \{\zeta, \mathrm{Li}\}$, put

$$\mathscr{S}_d^{\zeta}(s) = S_d(s), \qquad \mathscr{S}_d^{\mathrm{Li}}(s) = \frac{1}{L_d^s}.$$

Definition 2.1. Let $\bullet \in \{\zeta, \mathrm{Li}\}$. For any non-empty index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$, define

$$\mathscr{L}^{\bullet}_{\infty}([\mathfrak{s}]) = \sum_{d_1 > \cdots > d_r \geq 0} \prod_{i=1}^{r} \mathscr{S}^{\bullet}_{d_i}(s_i) \in \mathbb{C}_{\infty},$$

$$\mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\infty}([\mathfrak{s}]) = (-1)^r \sum_{0 \leq d_1 \leq \cdots \leq d_r} \prod_{i=1}^{r} \mathscr{S}^{\bullet}_{d_i}(s_i) \in \mathbb{C}_{\infty}.$$

Both maps take $1 = [\varnothing]$ to 1 and are extended R-linearly to $\mathfrak{H}_R$.

For every $\mathfrak{s} \in \mathcal{I}$, we have

$$\mathscr{L}^{\zeta}_{\infty}([\mathfrak{s}]) = \zeta_A(\mathfrak{s}), \qquad \mathscr{L}^{\zeta,\star-\mathrm{inv}}_{\infty}([\mathfrak{s}]) = \zeta^{\star-\mathrm{inv}}_A(\mathfrak{s}),$$

and

$$\mathscr{L}^{\mathrm{Li}}_{\infty}([\mathfrak{s}]) = \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}), \qquad \mathscr{L}^{\mathrm{Li},\star-\mathrm{inv}}_{\infty}([\mathfrak{s}]) = \mathrm{Li}^{\star-\mathrm{inv}}_{\mathfrak{s}}(\mathbf{1}).$$

2.2. **Shuffle and Stuffle Products.** We next recall the products corresponding to these two realizations. For the ζ-realization, the existence of the q-shuffle relations was established by Thakur [Tha10]. Chen subsequently obtained an explicit formula in depth one [Che15]. Based on Chen's formula, Yamamoto observed the recursive construction of the q-shuffle product, which was later formulated and proved by Shi (see [Shi18, Definition 3.1.3 and Theorem 3.1.4]). For the Li-realization, the corresponding product is the usual stuffle product [Cha14].

For $s, n, j \in \mathbb{N}$, set

$$\Delta^{[j]}_{s,n} = \begin{cases} (-1)^{s-1}\binom{j-1}{s-1} + (-1)^{n-1}\binom{j-1}{n-1}, & q-1 \mid j, \\ 0, & \text{otherwise.} \end{cases}$$

For any $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$, we put

$$\mathfrak{s}^+ = (s_1, \dots, s_{r-1}), \qquad \varnothing^+ = \varnothing, \qquad \mathfrak{s}^- = (s_2, \dots, s_r), \qquad \varnothing^- = \varnothing.$$

Definition 2.2. Let $\bullet \in \{\zeta, \mathrm{Li}\}$. We define two R-bilinear products on $\mathfrak{H}_R$, the q-shuffle product $*^{\zeta}$ and the stuffle product $*^{\mathrm{Li}}$, by

$$1 *^{\bullet} P = P *^{\bullet} 1 = P$$

and, for non-empty indices $\mathfrak{s} = (s_1, \dots, s_r)$ and $\mathfrak{n} = (n_1, \dots, n_\ell)$, by

$$\mathfrak{s} *^{\bullet} \mathfrak{n} = [s_1, \mathfrak{s}^- *^{\bullet} \mathfrak{n}] + [n_1, \mathfrak{s} *^{\bullet} \mathfrak{n}^-] + [s_1 + n_1, \mathfrak{s}^- *^{\bullet} \mathfrak{n}^-] + D^{\bullet}_{\mathfrak{s}}(\mathfrak{n}). \tag{1}$$

Here $D^{\mathrm{Li}}_{\mathfrak{s}}(\mathfrak{n}) = 0$, while

$$D^{\zeta}_{\mathfrak{s}}(\mathfrak{n}) = \sum_{j=1}^{s_1+n_1-1} \Delta^{[j]}_{s_1,n_1}[s_1 + n_1 - j, [j] *^{\zeta} (\mathfrak{s}^- *^{\zeta} \mathfrak{n}^-)]. \tag{2}$$

For every non-empty $\mathfrak{s} \in \mathcal{I}$, we set $D^{\bullet}_{\mathfrak{s}}(1) = 0$ and extend $D^{\bullet}_{\mathfrak{s}}(-)$ to $\mathfrak{H}_R$ R-linearly. We abbreviate

$$D^{\bullet}_{c}(-) = D^{\bullet}_{(c)}(-), \qquad D_{\mathfrak{s}}(-) = D^{\zeta}_{\mathfrak{s}}(-).$$

Thus (1) specializes to the q-shuffle product of ∞-adic multiple zeta values when $\bullet = \zeta$, and to the usual stuffle product when $\bullet = \mathrm{Li}$. Then we have the following theorem.

Theorem 2.3 ([Cha14, Shi18]; see also [CCM23, Proposition 2.7]). *Let* $\bullet \in \{\zeta, \mathrm{Li}\}$. *For any* $P, Q \in \mathfrak{H}_R$, *we have*

$$\mathscr{L}^{\bullet}_{\infty}(P *^{\bullet} Q) = \mathscr{L}^{\bullet}_{\infty}(P)\mathscr{L}^{\bullet}_{\infty}(Q).$$

2.3. **Linear Relations.** By [Cha14, Theorem 2.2.1], it is known that $\mathfrak{Z}_{\infty,R}$ forms a $\mathbb{Z}_{\geq 0}$-graded algebra and every $\overline{K}$-linear relation among $\zeta_A(\mathfrak{s})$ comes from a K-linear relation. Todd first conjectured a formula for the dimension of $\mathfrak{Z}_{\infty,K,w}$ in each weight w [Tod18, Conjecture 7.1]. Later, Thakur further conjectured that a basis of $\mathfrak{Z}_{\infty,K,w}$ is given by the values $\zeta_A(\mathfrak{s})$ with $\mathfrak{s}$ ranging over the set of Thakur indices of weight w [Tha17, Conjecture 8.2],

$$\mathcal{I}^{\mathrm{T}}_w = \begin{cases} \{\varnothing\}, & w = 0, \\ \{(s_1, \dots, s_r) \in \mathcal{I} : s_i \leq q \ (1 \leq i < r), s_r < q, \mathrm{wt}(\mathfrak{s}) = w\}, & w \geq 1. \end{cases}$$

We remark that $\zeta_A(\mathfrak{s}) = \mathscr{L}^{\zeta}_{\infty}([\mathfrak{s}]) = \mathscr{L}^{\mathrm{Li}}_{\infty}([\mathfrak{s}]) = \mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ for all $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}} = \bigcup_{w \geq 0} \mathcal{I}^{\mathrm{T}}_w$ (see [CCM23, (2.7)]). In [ND21, Theorem A], Ngo Dac proved that every multiple zeta value is an $\mathbb{F}_p[L_1]$-linear combination of these values. The full basis conjecture was later proved independently in [CCM23, Theorem 1.5] and [IKLNP24, Theorem B]. Precisely, we have the following theorem.

Theorem 2.4 ([CCM23, Theorem 1.5 and Corollary 3.7], [IKLNP24, Theorem B]; see also [Mis26, §1.2]). *Let* $\bullet \in \{\zeta, \mathrm{Li}\}$. *Then for each* $w \geq 0$,

$$\mathfrak{Z}_{\infty,R,w} = \mathscr{L}^{\bullet}_{\infty}(\mathfrak{H}_{R,w}),$$

and

$$\{\mathscr{L}^{\bullet}_{\infty}([\mathfrak{s}]) : \mathfrak{s} \in \mathcal{I}^{\mathrm{T}}_w\}$$

forms an R*-basis of* $\mathfrak{Z}_{\infty,R,w}$.

In fact, the linear relations among these values are well understood from [CCM23] (see also [IKLNP24]), and Mishiba considered the same framework over an arbitrary $\mathbb{F}_p[L_1]$-subalgebra R of $\overline{K}$ (see [Mis26, §2.3]). We recall the formulation below. Notice that we assume R is an $\mathbb{F}_p(L_1)$-algebra (see Remarks 4.3 and 4.5).

Definition 2.5 ([CCM23, §3]; see also [Mis26, §2.3]). For $c \in \mathbb{N}$, $\mathfrak{s} \in \mathcal{I}$, and $\bullet \in \{\zeta, \mathrm{Li}\}$, define the R-linear map

$$\alpha^{\bullet}_{c;\mathfrak{s}} : \mathfrak{H}_R \longrightarrow \mathfrak{H}_R, \qquad \alpha^{\bullet}_{c;\mathfrak{s}}(P) = [c, \mathfrak{s} *^{\bullet} P].$$

For $m \geq 1$, put

$$\alpha^{\bullet,m}_{c;\mathfrak{s}} = \underbrace{\alpha^{\bullet}_{c;\mathfrak{s}} \circ \cdots \circ \alpha^{\bullet}_{c;\mathfrak{s}}}_{m \text{ times}}, \qquad \alpha^{\bullet,0}_{c;\mathfrak{s}} = \mathrm{id}_{\mathfrak{H}_R}.$$

We also define the R-bilinear map

$$\boxplus : \mathfrak{H}_R \times \mathfrak{H}_R \longrightarrow \mathfrak{H}_R$$

by

$$1 \boxplus P = P \boxplus 1 = 0 \qquad \text{and} \qquad \mathfrak{s} \boxplus \mathfrak{n} = [s_1, \dots, s_{r-1}, s_r + n_1, n_2, \dots, n_\ell]$$

for any $P \in \mathfrak{H}_R$ and non-empty indices $\mathfrak{s} = (s_1, \dots, s_r), \mathfrak{n} = (n_1, \dots, n_\ell) \in \mathcal{I}$. Here 0 denotes the zero vector of $\mathfrak{H}_R$, rather than the empty word $1 = [\varnothing]$.

For $c \in \mathbb{N}$ and $m \geq 1$, we write $\{c\}^m = (c, \dots, c)$ for the index consisting of m copies of c and $\{c\}^0 = \varnothing$.

Definition 2.6 ([CCM23, §3]; see also [Mis26, §2.3]). Let $\bullet \in \{\zeta, \mathrm{Li}\}$. Put $\varepsilon_\zeta = 1$ and $\varepsilon_{\mathrm{Li}} = 0$. For $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ and $m \geq 1$, define

$$\begin{aligned}\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) = {} & [\mathfrak{s}, \{q\}^m, \mathfrak{n}] + [\mathfrak{s}, \{q\}^m \sqcup\!\sqcup \mathfrak{n}] + \varepsilon_\bullet[\mathfrak{s}, \{q\}^{m-1}, D_q(\mathfrak{n})] \\ & - L_1^m[\mathfrak{s}, \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})] - L_1^m[\mathfrak{s}] \sqcup\!\sqcup \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}) \\ & - \varepsilon_\bullet L_1^m[\mathfrak{s}^+, D_{s_r}(\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}))],\end{aligned}$$

if $\mathfrak{s} = (s_1, \dots, s_r)$ is non-empty, and

$$\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n}) = [\{q\}^m, \mathfrak{n}] + [\{q\}^m \sqcup\!\sqcup \mathfrak{n}] + \varepsilon_\bullet[\{q\}^{m-1}, D_q(\mathfrak{n})] - L_1^m \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}).$$

We also set

$$\mathscr{R}^\bullet_R = \mathrm{span}_R \{\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) : \mathfrak{s}, \mathfrak{n} \in \mathcal{I},\ m \geq 1\} \subseteq \mathfrak{H}_R$$

and

$$\mathscr{R}^\bullet_{R,w} = \mathrm{span}_R \{\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) : \mathfrak{s}, \mathfrak{n} \in \mathcal{I},\ m \geq 1,\ \mathrm{wt}(\mathfrak{s}) + mq + \mathrm{wt}(\mathfrak{n}) = w\} \subseteq \mathfrak{H}_{R,w}$$

Then we have the following proposition.

Proposition 2.7 ([CCM23, §3]; see also [Mis26, Proposition 2.3.2]). *For every $\bullet \in \{\zeta, \mathrm{Li}\}$, $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$, and $m \geq 1$, we have*

$$\mathscr{L}^\bullet_\infty(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})) = 0.$$

Moreover, for all $w \geq 0$,

$$\ker\left(\mathscr{L}^\bullet_\infty|_{\mathfrak{H}_{R,w}} : \mathfrak{H}_{R,w} \longrightarrow \mathfrak{Z}_{\infty,R}\right) = \mathscr{R}^\bullet_{R,w}, \qquad \ker\left(\mathscr{L}^\bullet_\infty : \mathfrak{H}_R \longrightarrow \mathfrak{Z}_{\infty,R}\right) = \mathscr{R}^\bullet_R.$$

3. Evaluation Maps and Deconcatenation

In this section, we study evaluation maps together with the deconcatenation map. We first compare the ordinary and $\star$-inverse evaluations. We then study the behavior of the relations $\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})$ under deconcatenation. These results will be used in the next section to construct the coaction on ∞-adic multiple zeta values.

3.1. **Evaluation Maps and the $\star$-Inverse.** Throughout this section, let $\bullet \in \{\zeta, \mathrm{Li}\}$, let B be a commutative unital R-algebra, and let $\mathrm{ev}, \mathrm{ev}^{\star-\mathrm{inv}} : \mathfrak{H}_R \to B$ be R-linear maps. Put $\varepsilon_\zeta = 1$ and $\varepsilon_{\mathrm{Li}} = 0$. We consider the following conditions:

(C1) $\mathrm{ev}(1) = \mathrm{ev}^{\star-\mathrm{inv}}(1) = 1$.

(C2) For any non-empty index $\mathfrak{s} \in \mathcal{I}$,

$$m_B(\mathrm{ev} \otimes \mathrm{ev}^{\star-\mathrm{inv}})\Delta([\mathfrak{s}]) = 0 = m_B(\mathrm{ev}^{\star-\mathrm{inv}} \otimes \mathrm{ev})\Delta([\mathfrak{s}]),$$

where m_B is the multiplication map of B.

(C3) When $\bullet = \zeta$, we have $\mathrm{ev}^{\star-\mathrm{inv}}([j]) = 0$ whenever $q - 1 \mid j$, while for $\bullet = \mathrm{Li}$, we have $\mathrm{ev}^{\star-\mathrm{inv}}([q-1]) = 0$.

(C4) $\mathrm{ev}(P *^\bullet Q) = \mathrm{ev}(P)\mathrm{ev}(Q)$ for all $P, Q \in \mathfrak{H}_R$.

(C5) $\mathrm{ev}^{\star-\mathrm{inv}}(P *^\bullet Q) = \mathrm{ev}^{\star-\mathrm{inv}}(P)\mathrm{ev}^{\star-\mathrm{inv}}(Q)$ for all $P, Q \in \mathfrak{H}_R$.

We then define

$$\mathcal{R} = (\mathrm{id} \otimes \mathrm{ev}^{\star-\mathrm{inv}})\Delta : \mathfrak{H}_R \longrightarrow \mathfrak{H}_R \otimes_R B$$

and equip $\mathfrak{H}_R \otimes_R B$ with the product

$$(P \otimes x) \star^\bullet (Q \otimes y) = (P *^\bullet Q) \otimes xy$$

for any $P, Q \in \mathfrak{H}_R$ and $x, y \in B$.

Lemma 3.1. *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$ be a non-empty index and $P \in \mathfrak{H}_R$. Then*

$$\mathcal{R}([\mathfrak{s}, P]) = 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}, P]) + \sum_{i=1}^{r-1} [\mathfrak{s}[:i]] \otimes \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[i+1:], P]) + ([\mathfrak{s}, -] \otimes \mathrm{id})\mathcal{R}(P),$$

where $[\mathfrak{s}, -]$ denotes left concatenation by $\mathfrak{s}$. In particular, if $\mathrm{dep}(\mathfrak{s}) = 1$, say $\mathfrak{s} = (s)$, then

$$\mathcal{R}([s, P]) = 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}([s, P]) + ([s, -] \otimes \mathrm{id})\mathcal{R}(P). \tag{3}$$

Proof. By R-linearity, it suffices to prove the assertion for $P = [\mathfrak{n}]$, where $\mathfrak{n} \in \mathcal{I}$. The result follows from the fact that

$$\mathcal{R}([\mathfrak{s}, \mathfrak{n}]) = 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}, \mathfrak{n}]) + \sum_{i=1}^{r-1} [\mathfrak{s}[:i]] \otimes \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[i+1:], \mathfrak{n}]) + \sum_{\mathfrak{n}=[A,B]} [\mathfrak{s}, A] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)$$

and $\mathcal{R}(\mathfrak{n}) = \sum_{\mathfrak{n}=[A,B]} A \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)$. □

Lemma 3.2. *For $\bullet = \mathrm{Li}$, assume Conditions C1, C2, and C4. For $\bullet = \zeta$, assume Conditions C1–C4. Then for any $P, Q \in \mathfrak{H}_R$, we have*

$$\mathcal{R}(P *^\bullet Q) = \mathcal{R}(P) \star^\bullet \mathcal{R}(Q).$$

In particular, Condition C5 holds.

Proof. By bilinearity, it is enough to take $P = [\mathfrak{s}]$ and $Q = [\mathfrak{n}]$, where $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$. We use induction on $\mathrm{dep}(\mathfrak{s}) + \mathrm{dep}(\mathfrak{n})$. If one of the indices is empty, the assertion follows immediately from Condition C1. For non-empty $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$, write $[\mathfrak{s}] = [s_1, \mathfrak{s}^-]$ and $[\mathfrak{n}] = [n_1, \mathfrak{n}^-]$. Recall that

$$\mathfrak{s} *^\bullet \mathfrak{n} = [s_1, \mathfrak{s}^- *^\bullet \mathfrak{n}] + [n_1, \mathfrak{s} *^\bullet \mathfrak{n}^-] + [s_1 + n_1, \mathfrak{s}^- *^\bullet \mathfrak{n}^-] + \varepsilon_\bullet \sum_{1 \le j < s_1 + n_1} \Delta^{[j]}_{s_1, n_1} [s_1 + n_1 - j, [j] *^\zeta (\mathfrak{s}^- *^\zeta \mathfrak{n}^-)].$$

Write

$$\mathcal{R}(\mathfrak{s}^-) = \sum_{\mathfrak{s}^-=[A,B]} A \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B), \qquad \mathcal{R}(\mathfrak{n}^-) = \sum_{\mathfrak{n}^-=[C,D]} C \otimes \mathrm{ev}^{\star-\mathrm{inv}}(D).$$

Then

$$\mathcal{R}(\mathfrak{s}) = 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}) + \sum_{\mathfrak{s}^-=[A,B]} [s_1, A] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B),$$

and similarly

$$\mathcal{R}(\mathfrak{n}) = 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{n}) + \sum_{\mathfrak{n}^-=[C,D]} [n_1, C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(D).$$

Hence, $\mathcal{R}(\mathfrak{s}) \star^\bullet \mathcal{R}(\mathfrak{n})$ consists of the term

$$1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s})\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{n}), \tag{4}$$

the two cross terms

$$\sum_{\mathfrak{s}^-=[A,B]} [s_1, A] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{n}), \tag{5}$$

and

$$\sum_{\mathfrak{n}^-=[C,D]} [n_1, C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s})\mathrm{ev}^{\star-\mathrm{inv}}(D), \tag{6}$$

together with

$$\sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} ([s_1, A] *^\bullet [n_1, C]) \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D). \tag{7}$$

Applying the recursion to the first tensor factor in (7), it becomes the sum of the following three terms:

$$\sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [s_1, A *^\bullet [n_1, C]] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D), \tag{8}$$

$$\sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [n_1, [s_1, A] *^\bullet C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D), \tag{9}$$

$$\sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [s_1 + n_1, A *^\bullet C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D), \tag{10}$$

together with

$$\varepsilon_\bullet \sum_{1 \le j < s_1+n_1} \Delta^{[j]}_{s_1,n_1} \sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [s_1 + n_1 - j, [j] *^\zeta (A *^\zeta C)] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D). \tag{11}$$

Notice that the sum of (5) and (8) becomes

$$\begin{aligned}
&\sum_{\mathfrak{s}^-=[A,B]} [s_1, A] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{n}) + \sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [s_1, A *^\bullet [n_1, C]] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D)\\
&= ([s_1, -] \otimes \mathrm{id}) \left(\mathcal{R}(\mathfrak{s}^-) \star^\bullet \mathcal{R}(\mathfrak{n})\right)\\
&= ([s_1, -] \otimes \mathrm{id})\mathcal{R}(\mathfrak{s}^- *^\bullet \mathfrak{n})\\
&= \mathcal{R}\left([s_1, \mathfrak{s}^- *^\bullet \mathfrak{n}]\right) - 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}\left([s_1, \mathfrak{s}^- *^\bullet \mathfrak{n}]\right),
\end{aligned}$$

where the second equality follows from the induction hypothesis and the last equality follows from (3). Similarly, the sum of (6) and (9) gives

$$\begin{aligned}
&\sum_{\mathfrak{n}^-=[C,D]} [n_1, C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s})\mathrm{ev}^{\star-\mathrm{inv}}(D) + \sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [n_1, [s_1, A] *^\bullet C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D)\\
&= ([n_1, -] \otimes \mathrm{id}) \left(\mathcal{R}(\mathfrak{s}) \star^\bullet \mathcal{R}(\mathfrak{n}^-)\right)\\
&= ([n_1, -] \otimes \mathrm{id})\mathcal{R}(\mathfrak{s} *^\bullet \mathfrak{n}^-)\\
&= \mathcal{R}\left([n_1, \mathfrak{s} *^\bullet \mathfrak{n}^-]\right) - 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}\left([n_1, \mathfrak{s} *^\bullet \mathfrak{n}^-]\right).
\end{aligned}$$

For (10), by (3) and the induction hypothesis, we have

$$\begin{aligned}
&\sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [s_1 + n_1, A *^\bullet C] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D)\\
&= ([s_1 + n_1, -] \otimes \mathrm{id}) \left(\mathcal{R}(\mathfrak{s}^-) \star^\bullet \mathcal{R}(\mathfrak{n}^-)\right)\\
&= ([s_1 + n_1, -] \otimes \mathrm{id})\mathcal{R}(\mathfrak{s}^- *^\bullet \mathfrak{n}^-)\\
&= \mathcal{R}\left([s_1 + n_1, \mathfrak{s}^- *^\bullet \mathfrak{n}^-]\right) - 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}\left([s_1 + n_1, \mathfrak{s}^- *^\bullet \mathfrak{n}^-]\right).
\end{aligned}$$

Now, if $\bullet = \zeta$, note that for every j with $\Delta^{[j]}_{s_1,n_1} \neq 0$, we have $q - 1 \mid j$. Hence, by Condition C3,

$$\mathcal{R}([j]) = 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}([j]) + [j] \otimes 1 = [j] \otimes 1.$$

Applying the induction hypothesis gives

$$\begin{aligned}
&\mathcal{R}\left([j] *^\zeta (\mathfrak{s}^- *^\zeta \mathfrak{n}^-)\right)\\
&= \mathcal{R}([j]) \star^\zeta \mathcal{R}(\mathfrak{s}^- *^\zeta \mathfrak{n}^-)\\
&= ([j] \otimes 1) \star^\zeta \left(\mathcal{R}(\mathfrak{s}^-) \star^\zeta \mathcal{R}(\mathfrak{n}^-)\right)\\
&= \sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} \left([j] *^\zeta (A *^\zeta C)\right) \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D).
\end{aligned}$$

Therefore (11) becomes

$$\begin{aligned}
&\sum_{1\le j<s_1+n_1} \Delta^{[j]}_{s_1,n_1} \sum_{\substack{\mathfrak{s}^-=[A,B]\\ \mathfrak{n}^-=[C,D]}} [s_1 + n_1 - j, [j] *^\zeta (A *^\zeta C)] \otimes \mathrm{ev}^{\star-\mathrm{inv}}(B)\mathrm{ev}^{\star-\mathrm{inv}}(D)\\
&= \sum_{1\le j<s_1+n_1} \Delta^{[j]}_{s_1,n_1} ([s_1 + n_1 - j, -] \otimes \mathrm{id})\mathcal{R}\left([j] *^\zeta (\mathfrak{s}^- *^\zeta \mathfrak{n}^-)\right)\\
&= \mathcal{R}\left(\sum_{1\le j<s_1+n_1} \Delta^{[j]}_{s_1,n_1} [s_1 + n_1 - j, [j] *^\zeta (\mathfrak{s}^- *^\zeta \mathfrak{n}^-)]\right)\\
&\quad - 1 \otimes \mathrm{ev}^{\star-\mathrm{inv}}\left(\sum_{1\le j<s_1+n_1} \Delta^{[j]}_{s_1,n_1} [s_1 + n_1 - j, [j] *^\zeta (\mathfrak{s}^- *^\zeta \mathfrak{n}^-)]\right),
\end{aligned}$$

where the last equality follows from (3).

Therefore, for either $\bullet \in \{\zeta, \mathrm{Li}\}$, we see that

$$\mathcal{R}(\mathfrak{s} *^\bullet \mathfrak{n}) - \mathcal{R}(\mathfrak{s}) \star^\bullet \mathcal{R}(\mathfrak{n}) = 1 \otimes \left(\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s} *^\bullet \mathfrak{n}) - \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s})\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{n})\right). \tag{12}$$

It remains to show that the right-hand side of (12) vanishes. Apply $m_B(\mathrm{ev}\otimes\mathrm{id})$ to both sides of (12). By Condition C2,

$$m_B(\mathrm{ev}\otimes\mathrm{id})\mathcal{R}(\mathfrak{s}*^\bullet\mathfrak{n})=0.$$

On the other hand, by Condition C4, $m_B(\mathrm{ev}\otimes\mathrm{id})$ is multiplicative with respect to $\star^\bullet$. Hence, Condition C2 implies that

$$m_B(\mathrm{ev}\otimes\mathrm{id})\left(\mathcal{R}(\mathfrak{s})\star^\bullet\mathcal{R}(\mathfrak{n})\right)=m_B(\mathrm{ev}\otimes\mathrm{id})\mathcal{R}(\mathfrak{s})\,m_B(\mathrm{ev}\otimes\mathrm{id})\mathcal{R}(\mathfrak{n})=0.$$

Therefore, applying $m_B(\mathrm{ev}\otimes\mathrm{id})$ to (12) gives Condition C5

$$\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}*^\bullet\mathfrak{n})=\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s})\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{n}).$$

Substituting this into (12), we obtain

$$\mathcal{R}(\mathfrak{s}*^\bullet\mathfrak{n})=\mathcal{R}(\mathfrak{s})\star^\bullet\mathcal{R}(\mathfrak{n}),$$

which completes the proof. □

Lemma 3.3. *For every $s\in\mathbb{N}$ and $d\geq 0$, fix $\mathscr{S}_d(s)\in B$, and fix $D\in\mathbb{N}\cup\{\infty\}$. Assume that the following sums are well-defined in B. Define the R-linear maps $\mathrm{ev},\mathrm{ev}^{\star-\mathrm{inv}}$ by $\mathrm{ev}(1)=\mathrm{ev}^{\star-\mathrm{inv}}(1)=1$ and, for every non-empty $\mathfrak{s}=(s_1,\dots,s_r)$, by*

$$\mathrm{ev}([\mathfrak{s}])=\sum_{D>d_1>\cdots>d_r\geq 0}\prod_{k=1}^{r}\mathscr{S}_{d_k}(s_k),$$

$$\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}])=(-1)^r\sum_{0\leq d_1\leq\cdots\leq d_r<D}\prod_{k=1}^{r}\mathscr{S}_{d_k}(s_k).$$

Then Conditions C1 and C2 hold.

Proof. Condition C1 follows immediately from $\mathrm{ev}(1)=\mathrm{ev}^{\star-\mathrm{inv}}(1)=1$. Let $\mathfrak{s}=(s_1,\dots,s_r)\in\mathcal{I}$ be non-empty. We first prove

$$\sum_{i=0}^{r}\mathrm{ev}(\mathfrak{s}[:i])\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:])=0.$$

Set $H_0=H_{r+1}=0$, and, for $1\leq i\leq r$, define

$$H_i=(-1)^{r-i}\sum_{\substack{D>d_1>\cdots>d_i\geq 0\\ d_i\leq d_{i+1}\leq\cdots\leq d_r<D}}\prod_{k=1}^{r}\mathscr{S}_{d_k}(s_k).$$

In particular,

$$H_1=-\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}),\qquad H_r=\mathrm{ev}(\mathfrak{s}).$$

We claim that, for every $0\leq i\leq r$,

$$\mathrm{ev}(\mathfrak{s}[:i])\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:])=H_i-H_{i+1}. \tag{13}$$

For $i=0$, this follows from

$$H_0-H_1=\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s})=\mathrm{ev}(\varnothing)\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}),$$

while for $i=r$,

$$H_r-H_{r+1}=\mathrm{ev}(\mathfrak{s})=\mathrm{ev}(\mathfrak{s})\mathrm{ev}^{\star-\mathrm{inv}}(\varnothing).$$

For $1 \leq i \leq r-1$, we have

$$\mathrm{ev}(\mathfrak{s}[:i])\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]) = (-1)^{r-i} \sum_{\substack{D>d_1>\cdots>d_i\geq 0 \\ 0\leq d_{i+1}\leq\cdots\leq d_r<D}} \prod_{k=1}^{r} \mathscr{S}_{d_k}(s_k).$$

There is no condition relating d_i and d_{i+1} in this sum. Splitting it into the two disjoint cases

$$d_i \leq d_{i+1} \qquad \text{and} \qquad d_i > d_{i+1},$$

gives

$$\mathrm{ev}(\mathfrak{s}[:i])\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]) = H_i - H_{i+1}.$$

Thus (13) holds for every $0 \leq i \leq r$. Summing over i yields

$$\sum_{i=0}^{r} \mathrm{ev}(\mathfrak{s}[:i])\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]) = \sum_{i=0}^{r}(H_i - H_{i+1}) = H_0 - H_{r+1} = 0.$$

Similarly,

$$\sum_{i=0}^{r} \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[:i])\mathrm{ev}(\mathfrak{s}[i+1:]) = 0.$$

Thus, Condition C2 holds. □

Lemma 3.4. *Let $\bullet \in \{\zeta, \mathrm{Li}\}$. Assume Conditions C1 and C2. If Condition C4 holds, then*

$$\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}_R) \subseteq \mathrm{ev}(\mathfrak{H}_R).$$

If Condition C5 holds, then

$$\mathrm{ev}(\mathfrak{H}_R) \subseteq \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}_R).$$

In particular, if both Conditions C4 and C5 hold, then

$$\mathrm{ev}(\mathfrak{H}_R) = \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}_R).$$

Proof. Suppose that Condition C4 holds. We prove $\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}) \in \mathrm{ev}(\mathfrak{H}_R)$ by induction on $\mathrm{dep}(\mathfrak{s})$. The case $\mathfrak{s} = \varnothing$ follows from Condition C1. For any non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, Condition C2 gives

$$\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}) = -\sum_{i=1}^{r} \mathrm{ev}(\mathfrak{s}[:i])\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]).$$

For every i, the induction hypothesis gives $P_i \in \mathfrak{H}_R$ such that

$$\mathrm{ev}(P_i) = \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]).$$

Hence, Condition C4 yields

$$\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{s}) = \mathrm{ev}\left(-\sum_{i=1}^{r} [\mathfrak{s}[:i]] *^{\bullet} P_i\right) \in \mathrm{ev}(\mathfrak{H}_R).$$

By R-linearity, this proves $\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}_R) \subseteq \mathrm{ev}(\mathfrak{H}_R)$. On the other hand, if Condition C5 holds, the same argument shows that $\mathrm{ev}(\mathfrak{H}_R) \subseteq \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}_R)$. This completes the proof. □

3.2. **Deconcatenation of the Linear Relations.** We next formulate a consequence of the preceding deconcatenation formalism which will be used for both realizations. Let $J \subseteq B$ be an ideal and write

$$\pi : B \longrightarrow \overline{B}$$

for the quotient map where $\overline{B} = B/J$. Put

$$\overline{\mathrm{ev}} = \pi \circ \mathrm{ev}, \qquad \overline{\mathrm{ev}}^{\star-\mathrm{inv}} = \pi \circ \mathrm{ev}^{\star-\mathrm{inv}}.$$

We assume Conditions C1, C2, and C4, and in addition assume that

$$\begin{cases} \overline{\mathrm{ev}}([j]) = 0 & \text{whenever } q-1 \mid j, \text{ if } \bullet = \zeta, \\ \overline{\mathrm{ev}}([q-1]) = 0 & \text{if } \bullet = \mathrm{Li}. \end{cases} \tag{14}$$

The maps $\overline{\mathrm{ev}}$ and $\overline{\mathrm{ev}}^{\star-\mathrm{inv}}$ satisfy Conditions C1 and C2. Moreover, for every $j \in \mathbb{N}$, Condition C2 in depth one gives

$$\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([j]) = -\overline{\mathrm{ev}}([j]).$$

Hence (14) implies Condition C3 for the pair $(\overline{\mathrm{ev}}, \overline{\mathrm{ev}}^{\star-\mathrm{inv}})$. Applying Lemma 3.2 to this pair shows that $\overline{\mathrm{ev}}^{\star-\mathrm{inv}}$ is an R-algebra homomorphism with respect to $*^\bullet$. We may therefore apply Lemma 3.2 again to the pair $(\overline{\mathrm{ev}}^{\star-\mathrm{inv}}, \overline{\mathrm{ev}})$. If we put

$$\mathcal{R}_\pi = (\mathrm{id} \otimes \overline{\mathrm{ev}})\Delta : \mathfrak{H}_R \longrightarrow \mathfrak{H}_R \otimes_R \overline{B}$$

and equip $\mathfrak{H}_R \otimes_R \overline{B}$ with the product

$$(P \otimes x) \star^\bullet (Q \otimes y) = (P *^\bullet Q) \otimes xy,$$

then we obtain

$$\mathcal{R}_\pi(P *^\bullet Q) = \mathcal{R}_\pi(P) \star^\bullet \mathcal{R}_\pi(Q) \qquad (P, Q \in \mathfrak{H}_R). \tag{15}$$

For the remainder of this subsection, we abbreviate $\alpha = \alpha^\bullet_{1;(q-1)}$ (recall Definition 2.5). For every non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, we define the R-linear map

$$B^\bullet_{\mathfrak{s}} : \mathfrak{H}_R \longrightarrow \mathfrak{H}_R$$

by

$$B^\bullet_{\mathfrak{s}}(P) = [\mathfrak{s}, P] + \mathfrak{s} \uplus P + \varepsilon_\bullet[\mathfrak{s}^+, D_{s_r}(P)] \qquad (P \in \mathfrak{H}_R). \tag{16}$$

Thus $B^\bullet_{\mathfrak{s}}(1) = [\mathfrak{s}]$. For $m \geq 1$ and $\mathfrak{n} \in \mathcal{I}$, put

$$P^\bullet_m(\mathfrak{n}) = [\{q\}^m, \mathfrak{n}] + \{q\}^m \uplus \mathfrak{n} + \varepsilon_\bullet[\{q\}^{m-1}, D_q(\mathfrak{n})]. \tag{17}$$

Equivalently,

$$P^\bullet_m(\mathfrak{n}) = B^\bullet_{\{q\}^m}(\mathfrak{n}).$$

With this notation, we have (recall Definition 2.6)

$$\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) = [\mathfrak{s}, P^\bullet_m(\mathfrak{n})] - L^m_1 B^\bullet_{\mathfrak{s}}(\alpha^m(\mathfrak{n})) \tag{18}$$

for non-empty $\mathfrak{s} \in \mathcal{I}$ and

$$\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n}) = P^\bullet_m(\mathfrak{n}) - L^m_1 \alpha^m(\mathfrak{n}). \tag{19}$$

We first establish two auxiliary formulas.

Lemma 3.5. *Under the assumptions above, let $\mathfrak{n} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{n}) = \ell$. Then, for every $m \geq 0$,*

$$\begin{aligned}\mathcal{R}_\pi\left(\alpha^m(\mathfrak{n})\right) = 1 \otimes \overline{\mathrm{ev}}\left(\alpha^m(\mathfrak{n})\right) + \sum_{i=1}^{m} \alpha^i(1) \otimes \overline{\mathrm{ev}}\left(\alpha^{m-i}(\mathfrak{n})\right) \\ + \sum_{k=1}^{\ell} \alpha^m(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]).\end{aligned} \tag{20}$$

Moreover, for every $c \in \mathbb{N}$,

$$\varepsilon_\bullet \mathcal{R}_\pi(D_c(\mathfrak{n})) = \varepsilon_\bullet \left(1 \otimes \overline{\mathrm{ev}}(D_c(\mathfrak{n})) + \sum_{k=1}^{\ell} D_c(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]) \right). \tag{21}$$

Proof. By (14) and (15), we have

$$\mathcal{R}_\pi([q-1]) = [q-1] \otimes 1, \qquad \mathcal{R}_\pi([q-1] *^\bullet P) = ([q-1] \otimes 1) \star^\bullet \mathcal{R}_\pi(P)$$

for every $P \in \mathfrak{H}_R$. Applying (3) to the pair $(\overline{\mathrm{ev}}^{\star-\mathrm{inv}}, \overline{\mathrm{ev}})$ yields

$$\begin{aligned}\mathcal{R}_\pi(\alpha(P)) &= \mathcal{R}_\pi\left([1, [q-1] *^\bullet P]\right) \\ &= 1 \otimes \overline{\mathrm{ev}}(\alpha(P)) + ([1,-] \otimes \mathrm{id})\mathcal{R}_\pi([q-1] *^\bullet P) \\ &= 1 \otimes \overline{\mathrm{ev}}(\alpha(P)) + ([1,-] \otimes \mathrm{id})\left(([q-1] \otimes 1) \star^\bullet \mathcal{R}_\pi(P)\right) \\ &= 1 \otimes \overline{\mathrm{ev}}(\alpha(P)) + (\alpha \otimes \mathrm{id})\mathcal{R}_\pi(P). \qquad (22)\end{aligned}$$

We prove (20) by induction on m. For $m = 0$, the result is clear. Now suppose that the result holds for some $m \geq 0$. Then by (22),

$$\begin{aligned}\mathcal{R}_\pi(\alpha^{m+1}(\mathfrak{n})) &= 1 \otimes \overline{\mathrm{ev}}(\alpha^{m+1}(\mathfrak{n})) + (\alpha \otimes \mathrm{id})\mathcal{R}_\pi(\alpha^m(\mathfrak{n})) \\ &= 1 \otimes \overline{\mathrm{ev}}(\alpha^{m+1}(\mathfrak{n})) + \alpha(1) \otimes \overline{\mathrm{ev}}(\alpha^m(\mathfrak{n})) \\ &\quad + \sum_{i=1}^{m} \alpha^{i+1}(1) \otimes \overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n})) + \sum_{j=1}^{\ell} \alpha^{m+1}(\mathfrak{n}[:j]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[j+1:]) \\ &= 1 \otimes \overline{\mathrm{ev}}(\alpha^{m+1}(\mathfrak{n})) + \sum_{i=1}^{m+1} \alpha^i(1) \otimes \overline{\mathrm{ev}}(\alpha^{m+1-i}(\mathfrak{n})) \\ &\quad + \sum_{k=1}^{\ell} \alpha^{m+1}(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]).\end{aligned}$$

Next, we prove (21). If $\bullet = \mathrm{Li}$, then (21) is immediate from $\varepsilon_{\mathrm{Li}} = 0$. Suppose that $\bullet = \zeta$. The assertion is clear for $\mathfrak{n} = \varnothing$. If $\mathfrak{n} = (n_1, \ldots, n_\ell) \in \mathcal{I}$ is non-empty, then (2) gives

$$D_c(\mathfrak{n}) = \sum_{j=1}^{c+n_1-1} \Delta_{c,n_1}^{[j]} [c + n_1 - j, [j] *^\zeta \mathfrak{n}^-].$$

Whenever $\Delta_{c,n_1}^{[j]} \neq 0$, we have $q - 1 \mid j$, so (14) implies

$$\mathcal{R}_\pi([j]) = [j] \otimes 1.$$

Using (15), we obtain

$$\mathcal{R}_\pi([j] *^\zeta \mathfrak{n}^-) = \sum_{k=1}^{\ell} ([j] *^\zeta \mathfrak{n}[:k]^-) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]).$$

Applying (3) to each summand and summing over j proves (21). □

We now derive the corresponding formulas for $B^\bullet_{\mathfrak{s}}$ and $P^\bullet_m$.

Proposition 3.6. *Under the assumptions above, for every non-empty $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$ and every $P \in \mathfrak{H}_R$, we have*

$$\mathcal{R}_\pi(B^\bullet_{\mathfrak{s}}(P)) = 1 \otimes \overline{\mathrm{ev}}(B^\bullet_{\mathfrak{s}}(P)) + \sum_{j=1}^{r-1} [\mathfrak{s}[:j]] \otimes \overline{\mathrm{ev}}(B^\bullet_{\mathfrak{s}[j+1:]}(P)) + (B^\bullet_{\mathfrak{s}} \otimes \mathrm{id})\mathcal{R}_\pi(P). \tag{23}$$

In particular, for every $m \geq 1$ and $\mathfrak{n} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{n}) = \ell$, we have

$$\begin{aligned} \mathcal{R}_\pi(P^\bullet_m(\mathfrak{n})) = 1 \otimes \overline{\mathrm{ev}}(P^\bullet_m(\mathfrak{n})) + \sum_{i=1}^{m-1} [\{q\}^i] \otimes \overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n})) \\ + [\{q\}^m] \otimes \overline{\mathrm{ev}}(\mathfrak{n}) + \sum_{k=1}^{\ell} P^\bullet_m(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]). \end{aligned} \tag{24}$$

Proof. By R-linearity, it suffices to take $P = [\mathfrak{n}]$, where $\mathfrak{n} \in \mathcal{I}$ and $\mathrm{dep}(\mathfrak{n}) = \ell$.

When $\bullet = \mathrm{Li}$, we have $\varepsilon_\bullet = 0$, and (23) follows directly from the deconcatenation of $[\mathfrak{s}, \mathfrak{n}] + \mathfrak{s} \boxplus \mathfrak{n}$. Now suppose that $\bullet = \zeta$. By (21), we have

$$\mathcal{R}_\pi(D_{s_r}(\mathfrak{n})) = 1 \otimes \overline{\mathrm{ev}}(D_{s_r}(\mathfrak{n})) + \sum_{k=1}^{\ell} D_{s_r}(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]).$$

Hence, by Lemma 3.1,

$$\begin{aligned} \mathcal{R}_\pi\left([\mathfrak{s}^+, D_{s_r}(\mathfrak{n})]\right) &= 1 \otimes \overline{\mathrm{ev}}\left([\mathfrak{s}^+, D_{s_r}(\mathfrak{n})]\right) \\ &\quad + \sum_{j=1}^{r-1} [\mathfrak{s}[:j]] \otimes \overline{\mathrm{ev}}\left([\mathfrak{s}[j+1:]^+, D_{s_r}(\mathfrak{n})]\right) \\ &\quad + \sum_{k=1}^{\ell} [\mathfrak{s}^+, D_{s_r}(\mathfrak{n}[:k])] \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]). \end{aligned}$$

Combining this with the deconcatenation of $[\mathfrak{s}, \mathfrak{n}] + \mathfrak{s} \boxplus \mathfrak{n}$, we obtain

$$\begin{aligned} \mathcal{R}_\pi(B^\bullet_{\mathfrak{s}}(\mathfrak{n})) = 1 \otimes \overline{\mathrm{ev}}(B^\bullet_{\mathfrak{s}}(\mathfrak{n})) + \sum_{j=1}^{r-1} [\mathfrak{s}[:j]] \otimes \overline{\mathrm{ev}}\left(B^\bullet_{\mathfrak{s}[j+1:]}(\mathfrak{n})\right) \\ + [\mathfrak{s}] \otimes \overline{\mathrm{ev}}(\mathfrak{n}) + \sum_{k=1}^{\ell} B^\bullet_{\mathfrak{s}}(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]). \end{aligned} \tag{25}$$

Since

$$\mathcal{R}_\pi(\mathfrak{n}) = 1 \otimes \overline{\mathrm{ev}}(\mathfrak{n}) + \sum_{k=1}^{\ell} \mathfrak{n}[:k] \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:])$$

and $B^\bullet_{\mathfrak{s}}(1) = [\mathfrak{s}]$, the last two terms are equal to $(B^\bullet_{\mathfrak{s}} \otimes \mathrm{id})\mathcal{R}_\pi(\mathfrak{n})$. This proves (23).

Finally, let $m \geq 1$. Since $P^\bullet_m(\mathfrak{n}) = B^\bullet_{\{q\}^m}(\mathfrak{n})$, applying (25) with $\mathfrak{s} = \{q\}^m$ gives

$$\begin{aligned} \mathcal{R}_\pi(P^\bullet_m(\mathfrak{n})) = 1 \otimes \overline{\mathrm{ev}}(P^\bullet_m(\mathfrak{n})) + \sum_{i=1}^{m-1} [\{q\}^i] \otimes \overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n})) \\ + [\{q\}^m] \otimes \overline{\mathrm{ev}}(\mathfrak{n}) + \sum_{k=1}^{\ell} P^\bullet_m(\mathfrak{n}[:k]) \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]), \end{aligned}$$

which completes the proof. □

3.3. **The Formal Coaction.** We are now ready to construct the formal coaction.

Proposition 3.7. *Assume that the pair* $(\mathrm{ev}, \mathrm{ev}^{\star-\mathrm{inv}})$ *satisfies Conditions* C1, C2, *and* C4, *and that* (14) *holds. Suppose moreover that*

$$\ker(\mathrm{ev}) = \mathscr{R}_R^\bullet = \mathrm{span}_R \left\{ \mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) : \mathfrak{s}, \mathfrak{n} \in \mathcal{I},\ m \geq 1 \right\}.$$

Then the assignment

$$\begin{aligned} \widetilde{\Delta}_{\mathrm{ev}} : \mathrm{ev}(\mathfrak{H}_R) &\longrightarrow \mathrm{ev}(\mathfrak{H}_R) \otimes_R \overline{\mathrm{ev}}(\mathfrak{H}_R), \\ \mathrm{ev}(P) &\longmapsto (\mathrm{ev} \otimes \overline{\mathrm{ev}})\Delta(P) \end{aligned} \tag{26}$$

is a well-defined R-algebra homomorphism. In particular, for every index $\mathfrak{s} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{s}) = r$,

$$\widetilde{\Delta}_{\mathrm{ev}}(\mathrm{ev}([\mathfrak{s}])) = \sum_{i=0}^{r} \mathrm{ev}([\mathfrak{s}[:i]]) \otimes \overline{\mathrm{ev}}([\mathfrak{s}[i+1:]]).$$

Proof. Note that we have

$$(\mathrm{ev} \otimes \overline{\mathrm{ev}})\Delta(P) = (\mathrm{ev} \otimes \mathrm{id})\mathcal{R}_\pi(P)$$

for any $P \in \mathfrak{H}_R$ and

$$\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})\right) = 0 \tag{27}$$

for all $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ and $m \geq 1$. We first prove the well-definedness. It suffices to check that

$$(\mathrm{ev} \otimes \mathrm{id})\mathcal{R}_\pi\left(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})\right) = 0 \tag{28}$$

for all $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ and $m \geq 1$.

Suppose first that $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$ is non-empty, and write $\mathrm{dep}(\mathfrak{n}) = \ell$. By (18),

$$\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) = [\mathfrak{s}, P_m^\bullet(\mathfrak{n})] - L_1^m B_{\mathfrak{s}}^\bullet\left(\alpha^m(\mathfrak{n})\right).$$

We compute the two terms separately. By Lemma 3.1 and (24),

$$\begin{aligned} \mathcal{R}_\pi\left([\mathfrak{s}, P_m^\bullet(\mathfrak{n})]\right) = {} & 1 \otimes \overline{\mathrm{ev}}\left([\mathfrak{s}, P_m^\bullet(\mathfrak{n})]\right) \\ & + \sum_{j=1}^{r-1} [\mathfrak{s}[:j]] \otimes \overline{\mathrm{ev}}\left([\mathfrak{s}[j+1:], P_m^\bullet(\mathfrak{n})]\right) \\ & + [\mathfrak{s}] \otimes \overline{\mathrm{ev}}(P_m^\bullet(\mathfrak{n})) \\ & + \sum_{i=1}^{m-1} [\mathfrak{s}, \{q\}^i] \otimes \overline{\mathrm{ev}}(P_{m-i}^\bullet(\mathfrak{n})) \\ & + [\mathfrak{s}, \{q\}^m] \otimes \overline{\mathrm{ev}}(\mathfrak{n}) \\ & + \sum_{k=1}^{\ell} [\mathfrak{s}, P_m^\bullet(\mathfrak{n}[:k])] \otimes \overline{\mathrm{ev}}(\mathfrak{n}[k+1:]). \end{aligned}$$

On the other hand, by (23) and (20),

$$\begin{aligned}\mathcal{R}_\pi\left(B^\bullet_{\mathfrak{s}}(\alpha^m(\mathfrak{n}))\right) &= 1\otimes\overline{\mathrm{ev}}\left(B^\bullet_{\mathfrak{s}}(\alpha^m(\mathfrak{n}))\right)\\ &\quad+\sum_{j=1}^{r-1}[\mathfrak{s}[:j]]\otimes\overline{\mathrm{ev}}\left(B^\bullet_{\mathfrak{s}[j+1:]}(\alpha^m(\mathfrak{n}))\right)\\ &\quad+[\mathfrak{s}]\otimes\overline{\mathrm{ev}}(\alpha^m(\mathfrak{n}))\\ &\quad+\sum_{i=1}^{m}B^\bullet_{\mathfrak{s}}(\alpha^i(1))\otimes\overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n}))\\ &\quad+\sum_{k=1}^{\ell}B^\bullet_{\mathfrak{s}}(\alpha^m(\mathfrak{n}[:k]))\otimes\overline{\mathrm{ev}}(\mathfrak{n}[k+1:]),\end{aligned}$$

where we have used $B^\bullet_{\mathfrak{s}}(1)=[\mathfrak{s}]$.

Subtracting L_1^m times the second identity from the first and applying $\mathrm{ev}\otimes\mathrm{id}$, we obtain

$$\begin{aligned}&(\mathrm{ev}\otimes\mathrm{id})\mathcal{R}_\pi\left(\mathscr{A}^\bullet(\mathfrak{s};m;\mathfrak{n})\right)\\ &=1\otimes\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\mathfrak{s};m;\mathfrak{n})\right)\\ &\quad+\sum_{j=1}^{r-1}\mathrm{ev}(\mathfrak{s}[:j])\otimes\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\mathfrak{s}[j+1:];m;\mathfrak{n})\right)\\ &\quad+\mathrm{ev}(\mathfrak{s})\otimes\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\varnothing;m;\mathfrak{n})\right)\\ &\quad+\sum_{i=1}^{m-1}\left(\mathrm{ev}([\mathfrak{s},\{q\}^i])\otimes\overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n}))-L_1^m\mathrm{ev}(B^\bullet_{\mathfrak{s}}(\alpha^i(1)))\otimes\overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n}))\right)\\ &\quad+\left(\mathrm{ev}([\mathfrak{s},\{q\}^m])-L_1^m\mathrm{ev}(B^\bullet_{\mathfrak{s}}(\alpha^m(1)))\right)\otimes\overline{\mathrm{ev}}(\mathfrak{n})\\ &\quad+\sum_{k=1}^{\ell}\mathrm{ev}\left(\mathscr{A}^\bullet(\mathfrak{s};m;\mathfrak{n}[:k])\right)\otimes\overline{\mathrm{ev}}(\mathfrak{n}[k+1:]).\end{aligned}\tag{29}$$

The first three terms in (29) vanish by (27). The last sum also vanishes, since

$$\mathscr{A}^\bullet(\mathfrak{s};m;\mathfrak{n}[:k])\in\mathscr{R}^\bullet_R=\ker(\mathrm{ev}).$$

It remains to consider the terms involving the index i. For every $i\geq 1$, we have

$$\mathscr{A}^\bullet(\mathfrak{s};i;\varnothing)\in\mathscr{R}^\bullet_R=\ker(\mathrm{ev}).$$

Since $P^\bullet_i(\varnothing)=[\{q\}^i]$, (18) implies that

$$\mathrm{ev}([\mathfrak{s},\{q\}^i])=L_1^i\mathrm{ev}\left(B^\bullet_{\mathfrak{s}}(\alpha^i(1))\right).\tag{30}$$

For $i=m$, this shows that the fifth term in (29) is zero. For $1\leq i<m$, using (30), the corresponding summand becomes

$$\begin{aligned}&\mathrm{ev}([\mathfrak{s},\{q\}^i])\otimes\overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n}))-L_1^m\mathrm{ev}(B^\bullet_{\mathfrak{s}}(\alpha^i(1)))\otimes\overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n}))\\ &=\mathrm{ev}([\mathfrak{s},\{q\}^i])\otimes\left(\overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n}))-L_1^{m-i}\overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n}))\right)\\ &=\mathrm{ev}([\mathfrak{s},\{q\}^i])\otimes\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\varnothing;m-i;\mathfrak{n})\right)=0,\end{aligned}$$

where the last equality follows from (27). Therefore (28) holds whenever $\mathfrak{s}\in\mathcal{I}$ is non-empty.

We now consider the case $\mathfrak{s}=\varnothing$. By (19),

$$\mathscr{A}^\bullet(\varnothing;m;\mathfrak{n})=P^\bullet_m(\mathfrak{n})-L_1^m\alpha^m(\mathfrak{n}).$$

Using (20) and (24), and then applying $\mathrm{ev}\otimes\mathrm{id}$, we obtain

$$\begin{aligned}
&(\mathrm{ev}\otimes\mathrm{id})\mathcal{R}_\pi\left(\mathscr{A}^\bullet(\varnothing;m;\mathfrak{n})\right)\\
&=1\otimes\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\varnothing;m;\mathfrak{n})\right)\\
&\quad+\sum_{i=1}^{m-1}\Big(\mathrm{ev}([\{q\}^i])\otimes\overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n}))-L_1^m\mathrm{ev}(\alpha^i(1))\otimes\overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n}))\Big)\\
&\quad+\left(\mathrm{ev}([\{q\}^m])-L_1^m\mathrm{ev}(\alpha^m(1))\right)\otimes\overline{\mathrm{ev}}(\mathfrak{n})\\
&\quad+\sum_{k=1}^{\ell}\mathrm{ev}\left(\mathscr{A}^\bullet(\varnothing;m;\mathfrak{n}[:k])\right)\otimes\overline{\mathrm{ev}}(\mathfrak{n}[k+1:]).
\end{aligned}$$

The first term is zero by (27), while the last sum vanishes since

$$\mathscr{A}^\bullet(\varnothing;m;\mathfrak{n}[:k])\in\ker(\mathrm{ev}).$$

It remains to treat the terms involving i. For every $i\geq 1$, we have

$$\mathscr{A}^\bullet(\varnothing;i;\varnothing)\in\mathscr{R}^\bullet_R=\ker(\mathrm{ev}).$$

Since

$$\mathscr{A}^\bullet(\varnothing;i;\varnothing)=[\{q\}^i]-L_1^i\alpha^i(1),$$

it follows that

$$\mathrm{ev}([\{q\}^i])=L_1^i\mathrm{ev}(\alpha^i(1)).$$

For $i=m$, this shows that the third term in the above expansion is zero. For $1\leq i<m$, the corresponding summand becomes

$$\begin{aligned}
&\mathrm{ev}([\{q\}^i])\otimes\overline{\mathrm{ev}}(P^\bullet_{m-i}(\mathfrak{n}))-L_1^m\mathrm{ev}(\alpha^i(1))\otimes\overline{\mathrm{ev}}(\alpha^{m-i}(\mathfrak{n}))\\
&=\mathrm{ev}([\{q\}^i])\otimes\overline{\mathrm{ev}}\left(\mathscr{A}^\bullet(\varnothing;m-i;\mathfrak{n})\right)=0.
\end{aligned}$$

Therefore (28) also holds for $\mathfrak{s}=\varnothing$. Hence (28) holds for all $\mathfrak{s},\mathfrak{n}\in\mathcal{I}$ and $m\geq 1$, and the assignment (26) is well-defined.

Finally, by Condition C4 and (15), for any $P,Q\in\mathfrak{H}_R$, we have

$$\begin{aligned}
\widetilde{\Delta}_{\mathrm{ev}}\left(\mathrm{ev}(P)\mathrm{ev}(Q)\right)&=(\mathrm{ev}\otimes\mathrm{id})\mathcal{R}_\pi(P*^\bullet Q)\\
&=(\mathrm{ev}\otimes\mathrm{id})\left(\mathcal{R}_\pi(P)\star^\bullet\mathcal{R}_\pi(Q)\right)\\
&=\widetilde{\Delta}_{\mathrm{ev}}(\mathrm{ev}(P))\widetilde{\Delta}_{\mathrm{ev}}(\mathrm{ev}(Q)).
\end{aligned}$$

Moreover, $\widetilde{\Delta}_{\mathrm{ev}}(1)=1\otimes 1$. Thus, $\widetilde{\Delta}_{\mathrm{ev}}$ is an R-algebra homomorphism. □

We next give the corresponding formula for the $\star$-inverse realization.

Proposition 3.8. *Under the assumptions of Proposition 3.7, for every index $\mathfrak{s}\in\mathcal{I}$ with $\mathrm{dep}(\mathfrak{s})=r$, we have*

$$\widetilde{\Delta}_{\mathrm{ev}}\left(\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}])\right)=\sum_{i=0}^{r}\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[i+1:]])\otimes\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([\mathfrak{s}[:i]]).$$

Proof. By Lemma 3.4, we have

$$\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}_R)\subseteq\mathrm{ev}(\mathfrak{H}_R),$$

so the left-hand side is well-defined. We prove the formula by induction on r. The case $r = 0$ is immediate. Let $r \geq 1$. By Condition C2,

$$\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}]) = -\sum_{i=1}^{r} \mathrm{ev}([\mathfrak{s}[:i]])\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[i+1:]]). \tag{31}$$

Since $\widetilde{\Delta}_{\mathrm{ev}}$ is an R-algebra homomorphism, Proposition 3.7 and the induction hypothesis imply that

$$\begin{aligned}\widetilde{\Delta}_{\mathrm{ev}}\left(\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}])\right) = -\sum_{i=1}^{r}\left(\sum_{j=0}^{i}\mathrm{ev}([\mathfrak{s}[:j]])\otimes\overline{\mathrm{ev}}([s_{j+1},\ldots,s_i])\right)\\ \cdot\left(\sum_{k=i}^{r}\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]])\otimes\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k])\right).\end{aligned}$$

Expanding the product gives

$$\begin{aligned}\widetilde{\Delta}_{\mathrm{ev}}\left(\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}])\right) = &-\sum_{1\leq i\leq k\leq r}\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]])\otimes\overline{\mathrm{ev}}([\mathfrak{s}[:i]])\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k])\\ &-\sum_{1\leq j\leq i\leq k\leq r}\mathrm{ev}([\mathfrak{s}[:j]])\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]])\otimes\overline{\mathrm{ev}}([s_{j+1},\ldots,s_i])\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k]).\end{aligned}$$

The first sum consists of the terms with $j = 0$, while the second contains those with $j \geq 1$.

For the first sum, we sum first over k. It becomes

$$-\sum_{k=1}^{r}\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]])\otimes\left(\sum_{i=1}^{k-1}\overline{\mathrm{ev}}([\mathfrak{s}[:i]])\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k])+\overline{\mathrm{ev}}([\mathfrak{s}[:k]])\right).$$

By Condition C2 applied to the index $\mathfrak{s}[:k]$, we have

$$\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([\mathfrak{s}[:k]])+\sum_{i=1}^{k-1}\overline{\mathrm{ev}}([\mathfrak{s}[:i]])\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k])+\overline{\mathrm{ev}}([\mathfrak{s}[:k]])=0.$$

Hence the first sum is equal to

$$\sum_{k=1}^{r}\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]])\otimes\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([\mathfrak{s}[:k]]).$$

We next consider the second sum. Separating the terms with $j = k$, and then summing first over i in the remaining terms, we obtain

$$\begin{aligned}&-\sum_{1\leq j<k\leq r}\mathrm{ev}([\mathfrak{s}[:j]])\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]])\\ &\otimes\left(\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{j+1},\ldots,s_k])+\sum_{i=j+1}^{k-1}\overline{\mathrm{ev}}([s_{j+1},\ldots,s_i])\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k])+\overline{\mathrm{ev}}([s_{j+1},\ldots,s_k])\right)\\ &\quad-\sum_{j=1}^{r}\mathrm{ev}([\mathfrak{s}[:j]])\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[j+1:]])\otimes 1.\end{aligned}$$

For $j < k$, Condition C2 applied to the index $(s_{j+1},\ldots,s_k)$ shows that

$$\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{j+1},\ldots,s_k])+\sum_{i=j+1}^{k-1}\overline{\mathrm{ev}}([s_{j+1},\ldots,s_i])\overline{\mathrm{ev}}^{\star-\mathrm{inv}}([s_{i+1},\ldots,s_k])+\overline{\mathrm{ev}}([s_{j+1},\ldots,s_k])=0.$$

Therefore, the second sum reduces to

$$-\sum_{j=1}^{r} \mathrm{ev}([\mathfrak{s}[:j]])\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[j+1:]]) \otimes 1 = \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}]) \otimes 1,$$

where the last equality follows from (31).

Combining the two sums, we obtain

$$\begin{aligned}\widetilde{\Delta}_{\mathrm{ev}}\left(\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}])\right) &= \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}]) \otimes 1 + \sum_{k=1}^{r} \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]]) \otimes \overline{\mathrm{ev}}^{\star-\mathrm{inv}}([\mathfrak{s}[:k]]) \\ &= \sum_{k=0}^{r} \mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}[k+1:]]) \otimes \overline{\mathrm{ev}}^{\star-\mathrm{inv}}([\mathfrak{s}[:k]]),\end{aligned}$$

as desired. □

4. The ∞-adic Coaction and the ⋆-Inverse

In this section, we apply the formalism developed in §3 to the ∞-adic multiple zeta value realization. Recall that

$$\overline{\mathfrak{Z}}_{\infty,R} = \mathfrak{Z}_{\infty,R}/\zeta_A(q-1)\mathfrak{Z}_{\infty,R},$$

and

$$\pi_\infty : \mathfrak{Z}_{\infty,R} \longrightarrow \overline{\mathfrak{Z}}_{\infty,R}$$

denotes the quotient map.

4.1. The ∞-adic Coaction.

We first verify the vanishing condition required in §3.3.

Lemma 4.1. *Let $j \in \mathbb{N}$ with $q-1 \mid j$. Then there exists $c_j \in \mathbb{F}_p(L_1)^\times$ such that*

$$\zeta_A(j) = c_j \zeta_A(q-1)^{j/(q-1)}.$$

In particular,

$$\pi_\infty(\zeta_A(j)) = 0.$$

Proof. By the Euler–Carlitz formula (see, for example, [Tha04, §5.2]),

$$\zeta_A(j) = \frac{\mathrm{BC}(j)}{\Gamma_{j+1}}\widetilde{\pi}^j, \qquad \zeta_A(q-1) = \frac{\widetilde{\pi}^{q-1}}{L_1},$$

where $\mathrm{BC}(j)$ is the j-th Bernoulli–Carlitz number, Γ_{j+1} is the $(j+1)$-st Carlitz factorial and $\widetilde{\pi}$ is a fixed Carlitz period (see [Tha04]). Writing $j = m(q-1)$, we obtain

$$\zeta_A(j) = \frac{\mathrm{BC}(j)L_1^m}{\Gamma_{j+1}}\zeta_A(q-1)^m.$$

Thus it remains to check that

$$\frac{\mathrm{BC}(j)L_1^m}{\Gamma_{j+1}} \in \mathbb{F}_p(L_1)^\times.$$

Recall that

$$\frac{z}{\exp_C(z)} = \sum_{n\geq 0} \frac{\mathrm{BC}(n)}{\Gamma_{n+1}} z^n$$

and

$$\exp_C(z) = \sum_{i\geq 0} \frac{z^{q^i}}{D_i},$$

where $D_0 = 1$ and $D_i = (\theta^{q^i} - \theta)D_{i-1}^q$ for all $i \geq 1$. Notice that we have

$$\theta^{q^i} - \theta = \sum_{r=0}^{i-1} (\theta^q - \theta)^{q^r} = -\sum_{r=0}^{i-1} L_1^{q^r} \in \mathbb{F}_p[L_1],$$

for every $i \geq 1$. We obtain that

$$\exp_C(z) \in \mathbb{F}_p(L_1)[\![z]\!], \qquad \frac{z}{\exp_C(z)} \in \mathbb{F}_p(L_1)[\![z]\!].$$

It follows that

$$\frac{\mathrm{BC}(j)}{\Gamma_{j+1}} \in \mathbb{F}_p(L_1),$$

and therefore

$$c_j = \frac{\mathrm{BC}(j)L_1^m}{\Gamma_{j+1}} \in \mathbb{F}_p(L_1)^\times.$$

The final assertion follows immediately. □

For $\bullet \in \{\zeta, \mathrm{Li}\}$, Conditions C1 and C2 for the pair

$$\left(\mathscr{L}_\infty^\bullet, \mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}\right)$$

follow from Lemma 3.3, while Condition C4 follows from Theorem 2.3. Hence Lemma 3.4 implies

$$\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}(\mathfrak{H}_R) \subseteq \mathscr{L}_\infty^\bullet(\mathfrak{H}_R) = \mathfrak{Z}_{\infty,R},$$

where the last equality for $\bullet = \mathrm{Li}$ follows from Theorem 2.4. We are now in a position to prove the following theorem.

Theorem 4.2. *There exists a unique R-algebra homomorphism*

$$\widetilde{\Delta}_\infty : \mathfrak{Z}_{\infty,R} \longrightarrow \mathfrak{Z}_{\infty,R} \otimes_R \overline{\mathfrak{Z}}_{\infty,R}$$

such that, for every $\bullet \in \{\zeta, \mathrm{Li}\}$ and every $\mathfrak{s} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{s}) = r$,

$$\widetilde{\Delta}_\infty\left(\mathscr{L}_\infty^\bullet([\mathfrak{s}])\right) = \sum_{i=0}^{r} \mathscr{L}_\infty^\bullet([\mathfrak{s}[:i]]) \otimes \pi_\infty\left(\mathscr{L}_\infty^\bullet([\mathfrak{s}[i+1:]])\right). \tag{32}$$

Moreover,

$$\widetilde{\Delta}_\infty\left(\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}])\right) = \sum_{i=0}^{r} \mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}[i+1:]]) \otimes \pi_\infty\left(\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}[:i]])\right). \tag{33}$$

In particular, $\widetilde{\Delta}_\infty$ endows $\mathfrak{Z}_{\infty,R}$ with a right $\overline{\mathfrak{Z}}_{\infty,R}$-comodule structure.

Proof. Fix $\bullet \in \{\zeta, \mathrm{Li}\}$. We apply Proposition 3.7 with

$$B = \mathfrak{Z}_{\infty,R}, \qquad J = \zeta_A(q-1)\mathfrak{Z}_{\infty,R},$$

and

$$\mathrm{ev} = \mathscr{L}_\infty^\bullet, \qquad \mathrm{ev}^{\star-\mathrm{inv}} = \mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}.$$

As observed above, Conditions C1, C2, and C4 hold. Moreover,

$$\ker(\mathscr{L}_\infty^\bullet) = \mathscr{R}_R^\bullet$$

by Proposition 2.7. If $\bullet = \zeta$, then Lemma 4.1 implies that for any $j \in \mathbb{N}$ with $q-1 \mid j$, we have

$$\pi_\infty(\mathscr{L}_\infty^\zeta([j])) = 0.$$

If $\bullet = \mathrm{Li}$, then

$$\mathscr{L}_\infty^{\mathrm{Li}}([q-1]) = \mathrm{Li}_{q-1}(1) = \zeta_A(q-1).$$

Hence, we have

$$\pi_\infty(\mathscr{L}_\infty^{\mathrm{Li}}([q-1])) = 0.$$

Thus (14) holds in both cases.

It follows from Proposition 3.7 that, for each $\bullet \in \{\zeta, \mathrm{Li}\}$, there exists a well-defined R-algebra homomorphism

$$\widetilde{\Delta}_\infty^\bullet : \mathfrak{Z}_{\infty,R} \longrightarrow \mathfrak{Z}_{\infty,R} \otimes_R \overline{\mathfrak{Z}}_{\infty,R}$$

such that

$$\widetilde{\Delta}_\infty^\bullet\left(\mathscr{L}_\infty^\bullet([\mathfrak{s}])\right) = \sum_{i=0}^{r} \mathscr{L}_\infty^\bullet([\mathfrak{s}[:i]]) \otimes \pi_\infty\left(\mathscr{L}_\infty^\bullet([\mathfrak{s}[i+1:]])\right).$$

We now show that

$$\widetilde{\Delta}_\infty^\zeta = \widetilde{\Delta}_\infty^{\mathrm{Li}}.$$

By Carlitz's formula (see [Tha04, Theorem 5.9.1]), for every $1 \leq s \leq q$ and $d \geq 0$, we have

$$\mathscr{S}_d^\zeta(s) = \mathscr{S}_d^{\mathrm{Li}}(s).$$

Hence the two coaction maps agree on

$$\mathscr{L}_\infty^\zeta([\mathfrak{s}]) = \mathscr{L}_\infty^{\mathrm{Li}}([\mathfrak{s}])$$

for every Thakur index $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}$. Since these elements form an R-basis of $\mathfrak{Z}_{\infty,R}$ by Theorem 2.4, the two coaction maps coincide on $\mathfrak{Z}_{\infty,R}$. We denote this common map by $\widetilde{\Delta}_\infty$ and the right $\overline{\mathfrak{Z}}_{\infty,R}$-comodule structure on $\mathfrak{Z}_{\infty,R}$ follows immediately from [Mis26, Theorem 1.3.4]. Finally, applying Proposition 3.8 to both realizations yields (33) and the uniqueness part is clear. □

Remark 4.3. The assumption that R is an $\mathbb{F}_p(L_1)$-algebra in Theorem 4.2 cannot in general be replaced by the assumption that R is an $\mathbb{F}_p[L_1]$-algebra. For example, let $q = 2$ and $R = \mathbb{F}_2[L_1]$. Expanding $\zeta_A(5)$ in the Thakur basis, we have

$$\zeta_A(5) = a\left(\zeta_A(1,1,1,1,1) + \zeta_A(1,1,2,1)\right) + b\left(\zeta_A(2,1,1,1) + \zeta_A(2,2,1)\right),$$

where

$$a = L_1^8 + L_1^5, \qquad b = L_1^8 + L_1^6 + L_1^5.$$

If the conclusion of Theorem 4.2 were valid over R, then, since (5) has depth one, the biweight $(2,3)$ component of $\widetilde{\Delta}_\infty(\zeta_A(5))$ would be zero. On the other hand, applying the coaction formula to the above expansion gives

$$(a + L_1 b)\,\zeta_A(1,1) \otimes \pi_\infty\left(\zeta_A(1,1,1) + \zeta_A(2,1)\right).$$

Here, we use $\zeta_A(2) = L_1\zeta_A(1,1)$. Moreover, note that

$$a + L_1 b = L_1^5\left(L_1^4 + L_1^3 + L_1^2 + L_1 + 1\right),$$

and

$$\zeta_A(1)\zeta_A(1,1) = (L_1+1)\left(\zeta_A(1,1,1) + \zeta_A(2,1)\right).$$

Since $L_1 + 1 \nmid a + L_1 b$ in $\mathbb{F}_2[L_1]$, the above biweight $(2,3)$ component is nonzero, a contradiction. Thus, Theorem 4.2 does not in general hold over an arbitrary $\mathbb{F}_p[L_1]$-subalgebra.

4.2. **The Coaction and the $\star$-Inverse.** By [Mis26, Theorem 1.3.4], Mishiba's coaction induces a coproduct

$$\Delta_\infty : \overline{\mathfrak{Z}}_{\infty,R} \longrightarrow \overline{\mathfrak{Z}}_{\infty,R} \otimes_R \overline{\mathfrak{Z}}_{\infty,R}$$

and an antipode

$$S_\infty : \overline{\mathfrak{Z}}_{\infty,R} \longrightarrow \overline{\mathfrak{Z}}_{\infty,R}.$$

We now determine the actions of Δ_∞ and S_∞ on the ∞-adic multiple zeta values.

Theorem 4.4. *For every $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$, we have*

$$\Delta_\infty\left(\pi_\infty(\zeta_A(\mathfrak{s}))\right) = \sum_{i=0}^{r} \pi_\infty(\zeta_A(\mathfrak{s}[:i])) \otimes \pi_\infty(\zeta_A(\mathfrak{s}[i+1:])) \tag{34}$$

and

$$S_\infty\left(\pi_\infty(\zeta_A(\mathfrak{s}))\right) = \pi_\infty\left(\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s})\right). \tag{35}$$

In particular, Conjecture 1.2 (ii) holds.

Proof. By the construction of Δ_∞ in [Mis26, Theorem 1.3.4], we have

$$\Delta_\infty \circ \pi_\infty = (\pi_\infty \otimes \mathrm{id}) \circ \widetilde{\Delta}_\infty.$$

Hence (34) follows immediately from (32) with $\bullet = \zeta$.

We prove (35) by induction on $\mathrm{dep}(\mathfrak{s})$. The case $\mathfrak{s} = \varnothing$ is clear. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$ be non-empty. By the antipode identity and (34),

$$S_\infty\left(\pi_\infty(\zeta_A(\mathfrak{s}))\right) = -\pi_\infty(\zeta_A(\mathfrak{s})) - \sum_{i=1}^{r-1} \pi_\infty(\zeta_A(\mathfrak{s}[:i]))S_\infty\left(\pi_\infty(\zeta_A(\mathfrak{s}[i+1:]))\right).$$

By the induction hypothesis, the right-hand side is

$$-\pi_\infty(\zeta_A(\mathfrak{s})) - \sum_{i=1}^{r-1} \pi_\infty(\zeta_A(\mathfrak{s}[:i]))\pi_\infty\left(\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:])\right).$$

On the other hand, Lemma 3.3 gives

$$\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s}) + \sum_{i=1}^{r-1} \zeta_A(\mathfrak{s}[:i])\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s}[i+1:]) + \zeta_A(\mathfrak{s}) = 0.$$

Applying π_∞ proves (35). □

Remark 4.5. The assumption that R is an $\mathbb{F}_p(L_1)$-algebra in Theorem 4.4 cannot in general be replaced by the assumption that R is an $\mathbb{F}_p[L_1]$-algebra. For example, let $q = 2$ and $R = \mathbb{F}_2[L_1]$. By expanding the relevant values in the Thakur basis, one obtains

$$S_\infty\left(\pi_\infty(\zeta_A(1,5))\right) - \pi_\infty\left(\zeta_A^{\star-\mathrm{inv}}(1,5)\right) = \pi_\infty(\zeta_A(6)) \neq 0.$$

Thus, the antipode formula (35) does not in general hold over an arbitrary $\mathbb{F}_p[L_1]$-subalgebra.

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