MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC: A KANEKO–ZAGIER-TYPE CONJECTURE

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ABSTRACT. In this paper, we study a comparison between ∞-adic and finite multiple zeta values in positive characteristic. We construct a surjective comparison map from the algebra of ∞-adic multiple zeta values to a quotient of the algebra of finite multiple zeta values. This map factors through the quotient by $\zeta_A(q-1)$, thereby establishing the well-definedness of a Kaneko–Zagier-type conjectural map. We further show that the comparison map is compatible with special values of Carlitz multiple polylogarithms, as well as with the corresponding $\star$-inverse values. As an application, we prove the upper bound predicted by Shi's dimension conjecture for finite multiple zeta values.

1. INTRODUCTION

1.1. **Classical Background.** We let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of integers. Throughout the paper, we adopt the convention that empty sums are zero and empty products are one. We let

$$\mathcal{I} = \{\varnothing\} \cup \bigcup_{r\geq 1} \mathbb{N}^r$$

denote the set of indices and define

$$\mathrm{dep}(\mathfrak{s}) = r, \qquad \mathrm{wt}(\mathfrak{s}) = s_1 + \cdots + s_r, \qquad \mathrm{dep}(\varnothing) = \mathrm{wt}(\varnothing) = 0$$

for any $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$. For an admissible non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, namely $s_1 \geq 2$, the classical *multiple zeta value* is defined by

$$\zeta(\mathfrak{s}) = \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}.$$

It is known that multiple zeta values admit the stuffle relations arising from their defining series, as well as the shuffle relations arising from their iterated integral representations. We put $\mathfrak{Z}$ for the $\mathbb{Q}$-algebra generated by all multiple zeta values.

An arithmetic counterpart of classical multiple zeta values is provided by *finite multiple zeta values* introduced by Kaneko and Zagier (see [KZ]; see also [Kan19]). Put

$$\mathcal{A} = \left(\prod_p \mathbb{Z}/(p)\right) \Big/ \left(\bigoplus_p \mathbb{Z}/(p)\right),$$

where p runs through the prime numbers. Note that there is a natural embedding $\mathbb{Q} \hookrightarrow \mathcal{A}$. For a non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, the corresponding finite multiple zeta value is

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defined by

$$\zeta_{\mathcal{A}}(\mathfrak{s}) = \left(\sum_{p>n_1>\cdots>n_r\geq 1} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} \bmod p \right)_p \in \mathcal{A}.$$

We write $\mathfrak{Z}_{\mathcal{A}}$ for the $\mathbb{Q}$-algebra generated by all finite multiple zeta values.

Kaneko and Zagier also introduced the *symmetric multiple zeta values* $\zeta_{\mathcal{S}}(\mathfrak{s})$, $\mathfrak{s} \in \mathcal{I}$, which are naturally regarded as elements of $\mathfrak{Z}/\zeta(2)\mathfrak{Z}$. They observed a striking similarity between the relations satisfied by finite and symmetric multiple zeta values and were led to the following Kaneko–Zagier conjecture [KZ].

Conjecture 1.1 (Kaneko, Zagier)**.** *There exists a well-defined* $\mathbb{Q}$*-algebra isomorphism*

$$\Phi_{\mathrm{KZ}} : \mathfrak{Z}_{\mathcal{A}} \xrightarrow{\sim} \mathfrak{Z}/\zeta(2)\mathfrak{Z}$$

such that for any $\mathfrak{s} \in \mathcal{I}$*,*

$$\Phi_{\mathrm{KZ}}(\zeta_{\mathcal{A}}(\mathfrak{s})) = \zeta_{\mathcal{S}}(\mathfrak{s}).$$

It is known by a theorem of Yasuda [Yas16] that the symmetric multiple zeta values span $\mathfrak{Z}/\zeta(2)\mathfrak{Z}$, so the conjectural map is automatically surjective once it is well-defined. A further connection between these two sides was found by Bachmann, Takeyama, and Tasaka [BTT18], who considered *finite multiple harmonic q-series* at roots of unity and showed that the real parts of their analytic limits yield symmetric multiple zeta values, whereas their algebraic reductions yield finite multiple zeta values, providing further evidence for the Kaneko–Zagier conjecture.

1.2. **Function Field Counterpart.** We now turn to the function field setting. Let p be a prime number and let q be a power of p. We let $A = \mathbb{F}_q[\theta]$ denote the polynomial ring over $\mathbb{F}_q$ in the variable θ, and let $K = \mathbb{F}_q(\theta)$ be its field of fractions. Let $K_\infty = \mathbb{F}_q((1/\theta))$ be the completion of K at the infinite place and $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞. We also let $\overline{K}$ denote the algebraic closure of K in $\mathbb{C}_\infty$.

Thakur introduced the function field analogue of multiple zeta values $\zeta_A(\mathfrak{s})$ in [Tha04]. Throughout this paper, we call them the *∞-adic multiple zeta values*. Chang [Cha14] later introduced the *Carlitz multiple polylogarithms*, whose special values $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ will also be considered below. Here $\mathbf{1}$ denotes the tuple $(1,\dots,1)$ of the appropriate depth. More recently, Mishiba [Mis26] introduced the corresponding $\star$-inverse values $\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s})$ and $\mathrm{Li}_{\mathfrak{s}}^{\star-\mathrm{inv}}(\mathbf{1})$. (We refer to §2.1 for the precise definitions.)

Put $L_1 = \theta - \theta^q$. We let $\mathfrak{Z}_\infty$ denote the K-algebra generated by the ∞-adic multiple zeta values $\zeta_A(\mathfrak{s})$, with $\mathfrak{s} \in \mathcal{I}$. For each $w \geq 0$, we also write

$$\mathfrak{Z}_{\infty,w} = \mathrm{span}_K \left\{ \zeta_A(\mathfrak{s}) : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w \right\}.$$

We next recall the *finite multiple zeta values over K*, denoted by $\zeta_{\mathcal{A}_K}(\mathfrak{s})$, which were studied in [CM17, Shi18]. Let v run through the monic irreducible polynomials of A, and put

$$\mathcal{A}_K = \left(\prod_v A/(v) \right) \Big/ \left(\bigoplus_v A/(v) \right).$$

There is a natural embedding $K \hookrightarrow \mathcal{A}_K$, and we regard $\mathcal{A}_K$ as a K-algebra. Chang and Mishiba introduced the corresponding special values of *finite Carlitz multiple polylogarithms*

$\mathrm{Li}_{\mathcal{A}_K,\mathfrak{s}}(\mathbf{1})$ in [CM17]. We will also use the corresponding $\star$-inverse values $\zeta^{\star-\mathrm{inv}}_{\mathcal{A}_K}(\mathfrak{s})$ and $\mathrm{Li}^{\star-\mathrm{inv}}_{\mathcal{A}_K,\mathfrak{s}}(\mathbf{1})$. (We refer to §2.1 for the precise definitions.)

We write $\mathfrak{Z}_{\mathcal{A}_K}$ for the K-algebra generated by all finite multiple zeta values $\zeta_{\mathcal{A}_K}(\mathfrak{s})$, with $\mathfrak{s} \in \mathcal{I}$. For $w \geq 0$, put

$$\mathfrak{Z}_{\mathcal{A}_K,w} = \mathrm{span}_K \left\{ \zeta_{\mathcal{A}_K}(\mathfrak{s}) : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w \right\}.$$

Recently, the author introduced in [Tsu26a] the *finite multiple harmonic u-series* using the Carlitz module. When evaluated at Carlitz torsion points, their analytic limits recover the ∞-adic multiple zeta values, while their algebraic reductions recover the finite multiple zeta values over K [Tsu26a, Theorems 3.18 and 3.19]. This provides a positive-characteristic analogue of the result of Bachmann, Takeyama, and Tasaka [BTT18], in which finite multiple harmonic q-series at roots of unity connect finite and symmetric multiple zeta values.

Motivated by the Kaneko–Zagier conjecture, the author proposed a closely related comparison in [Tsu26a, Question 3.24 and Remark 3.25]. We put

$$\mathfrak{g}_{\mathcal{A}_K} = \zeta_{\mathcal{A}_K}(q) - L_1 \zeta_{\mathcal{A}_K}(1, q-1) \in \mathfrak{Z}_{\mathcal{A}_K},$$

the finite element arising from Thakur's fundamental relation [Tha09, Theorem 5]. The corresponding comparison conjecture may be formulated as follows.

Conjecture 1.2 (cf. [Tsu26a, Question 3.24 and Remark 3.25])**.** *There exists a K-algebra isomorphism*

$$\Phi_{\infty,\mathcal{A}_K} : \mathfrak{Z}_\infty / \zeta_A(q-1)\mathfrak{Z}_\infty \xrightarrow{\sim} \mathfrak{Z}_{\mathcal{A}_K} / \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}$$

such that, for every $\mathfrak{s} \in \mathcal{I}$,

$$\Phi_{\infty,\mathcal{A}_K}\left(\zeta_A(\mathfrak{s}) \bmod \zeta_A(q-1)\mathfrak{Z}_\infty\right) = \zeta_{\mathcal{A}_K}(\mathfrak{s}) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

1.3. **Main Results.** We now state the main results of this paper. Our main result establishes a new comparison between ∞-adic and finite multiple zeta values. More precisely, we construct a well-defined surjective comparison map from the algebra of ∞-adic multiple zeta values to a quotient of the algebra of finite multiple zeta values. This map factors through the quotient by $\zeta_A(q-1)$, providing the well-definedness part for Conjecture 1.2.

Theorem 1.3 (Restated as Theorems 3.7 and 3.9)**.** *There exists a well-defined surjective K-algebra homomorphism*

$$\widetilde{\Phi}_{\infty,\mathcal{A}_K} : \mathfrak{Z}_\infty \longrightarrow \mathfrak{Z}_{\mathcal{A}_K} / \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}$$

such that, for every $\mathfrak{s} \in \mathcal{I}$,

$$\widetilde{\Phi}_{\infty,\mathcal{A}_K}\left(\zeta_A(\mathfrak{s})\right) = \zeta_{\mathcal{A}_K}(\mathfrak{s}) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

Moreover, the map is compatible with the corresponding special values of Carlitz multiple polylogarithms and with the $\star$-inverse values:

$$\begin{aligned}
\widetilde{\Phi}_{\infty,\mathcal{A}_K}\left(\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})\right) &= \mathrm{Li}_{\mathcal{A}_K,\mathfrak{s}}(\mathbf{1}) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}, \\
\widetilde{\Phi}_{\infty,\mathcal{A}_K}\left(\zeta_A^{\star-\mathrm{inv}}(\mathfrak{s})\right) &= \zeta^{\star-\mathrm{inv}}_{\mathcal{A}_K}(\mathfrak{s}) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}, \\
\widetilde{\Phi}_{\infty,\mathcal{A}_K}\left(\mathrm{Li}^{\star-\mathrm{inv}}_{\mathfrak{s}}(\mathbf{1})\right) &= \mathrm{Li}^{\star-\mathrm{inv}}_{\mathcal{A}_K,\mathfrak{s}}(\mathbf{1}) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.
\end{aligned}$$

Furthermore,

$$\zeta_A(q-1)\mathfrak{Z}_\infty \subseteq \ker \widetilde{\Phi}_{\infty,\mathcal{A}_K},$$

and hence $\widetilde{\Phi}_{\infty,\mathcal{A}_K}$ induces a surjective K-algebra homomorphism

$$\Phi_{\infty,\mathcal{A}_K} : \mathfrak{Z}_\infty/\zeta_A(q-1)\mathfrak{Z}_\infty \longrightarrow \mathfrak{Z}_{\mathcal{A}_K}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

The proof of the theorem is based on a comparison between the ∞-adic and finite realizations. More precisely, using the explicit generators for the kernel of the ∞-adic realization (see §2.3), we show that their finite realizations vanish modulo $\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}$ (see Lemma 3.5). This shows the well-definedness of the map. For the compatibility with the $\star$-inverse values, we use the formalism of evaluation maps in [Tsu26b, §3.1].

As an application of Theorem 1.3, we also obtain an upper bound for the dimension of $\mathfrak{Z}_{\mathcal{A}_K,w}$ (see Theorem 4.3).

The rest of the paper is organized as follows. In §2, we recall the necessary notation and background. In §3.1, we recall the formalism of evaluation maps that will be used in the sequel. In §3.2, we construct the comparison map from ∞-adic to finite multiple zeta values and prove its compatibility with the two realizations and their $\star$-inverse counterparts. Finally, in §4, we compare the dimension problems in the ∞-adic, v-adic, and finite settings, establish the finite dimension upper bound, and formulate a unified comparison conjecture.

2. Preliminaries

2.1. **Algebraic Setup.** We now fix the algebraic notation used throughout the paper. Recall that

$$\mathcal{I} = \{\varnothing\} \cup \bigcup_{r\geq 1} \mathbb{N}^r$$

denotes the set of indices. We let

$$\mathfrak{H} = \mathrm{span}_K\{[\mathfrak{s}] : \mathfrak{s} \in \mathcal{I}\}$$

denote the free K-vector space generated by the set of indices $\mathcal{I}$. For each $w \geq 0$, we also define

$$\mathfrak{H}_w = \mathrm{span}_K\{[\mathfrak{s}] : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w\}.$$

We put $1 = [\varnothing]$. Then the concatenation map

$$[-,-] : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$$

is the unique K-bilinear map determined by

$$[\mathfrak{s},\mathfrak{n}] = [s_1,\dots,s_r,n_1,\dots,n_\ell], \qquad [\varnothing,\mathfrak{s}] = [\mathfrak{s},\varnothing] = [\mathfrak{s}], \qquad [\varnothing,\varnothing] = [\varnothing],$$

for all non-empty indices $\mathfrak{s} = (s_1,\dots,s_r), \mathfrak{n} = (n_1,\dots,n_\ell) \in \mathcal{I}$. Iterated concatenations are understood K-multilinearly and are denoted by

$$[-,\dots,-] : \mathfrak{H} \times \cdots \times \mathfrak{H} \longrightarrow \mathfrak{H}.$$

We identify an index with its corresponding basis element when no confusion can arise.

For $\mathfrak{s} = (s_1,\dots,s_r) \in \mathcal{I}$ and $0 \leq i \leq r$, we write

$$\mathfrak{s}[:i] = (s_1,\dots,s_i), \qquad \mathfrak{s}[i+1:] = (s_{i+1},\dots,s_r),$$

with the convention that any truncation outside the indicated range is equal to $\varnothing$. We define the deconcatenation map

$$\Delta : \mathfrak{H} \longrightarrow \mathfrak{H} \otimes_K \mathfrak{H},$$

which is the unique K-linear map such that

$$\Delta([\mathfrak{s}]) = \sum_{i=0}^{r} [\mathfrak{s}[:i]] \otimes [\mathfrak{s}[i+1:]]$$

for all indices $\mathfrak{s} \in \mathcal{I}$ with $\mathrm{dep}(\mathfrak{s}) = r$.

We next define the realization maps used throughout the paper. For $d \geq 0$ and $s \in \mathbb{N}$, set

$$S_d(s) = \sum_{\substack{a \in A \text{ monic} \\ \deg a = d}} \frac{1}{a^s}, \qquad L_0 = 1, \qquad L_d = \prod_{i=1}^{d} (\theta - \theta^{q^i}) \quad (d \geq 1).$$

For $\bullet \in \{\zeta, \mathrm{Li}\}$, put

$$\mathscr{S}_d^{\zeta}(s) = S_d(s), \qquad \mathscr{S}_d^{\mathrm{Li}}(s) = \frac{1}{L_d^s}.$$

Definition 2.1. Let $\bullet \in \{\zeta, \mathrm{Li}\}$. For any non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, define

$$\mathscr{L}_{\infty}^{\bullet}([\mathfrak{s}]) = \sum_{d_1 > \cdots > d_r \geq 0} \prod_{i=1}^{r} \mathscr{S}_{d_i}^{\bullet}(s_i) \in K_{\infty},$$

$$\mathscr{L}_{\infty}^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}]) = (-1)^r \sum_{0 \leq d_1 \leq \cdots \leq d_r} \prod_{i=1}^{r} \mathscr{S}_{d_i}^{\bullet}(s_i) \in K_{\infty}.$$

Both maps take $1 = [\varnothing]$ to 1 and are extended K-linearly to $\mathfrak{H}$.

For every $\mathfrak{s} \in \mathcal{I}$, we have

$$\mathscr{L}_{\infty}^{\zeta}([\mathfrak{s}]) = \zeta_A(\mathfrak{s}), \qquad \mathscr{L}_{\infty}^{\zeta,\star-\mathrm{inv}}([\mathfrak{s}]) = \zeta_A^{\star-\mathrm{inv}}(\mathfrak{s}),$$

and

$$\mathscr{L}_{\infty}^{\mathrm{Li}}([\mathfrak{s}]) = \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}), \qquad \mathscr{L}_{\infty}^{\mathrm{Li},\star-\mathrm{inv}}([\mathfrak{s}]) = \mathrm{Li}_{\mathfrak{s}}^{\star-\mathrm{inv}}(\mathbf{1}).$$

Moreover, $\mathfrak{Z}_{\infty,w}$ is also spanned by $\mathscr{L}_{\infty}^{\mathrm{Li}}([\mathfrak{s}]) = \mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ with $\mathrm{wt}(\mathfrak{s}) = w$ (see [CCM23, Corollary 3.7]).

Definition 2.2. Let $\bullet \in \{\zeta, \mathrm{Li}\}$. For any non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, define

$$\mathscr{L}_{\mathcal{A}_K}^{\bullet}([\mathfrak{s}]) = \left(\sum_{\deg v > d_1 > \cdots > d_r \geq 0} \prod_{i=1}^{r} \mathscr{S}_{d_i}^{\bullet}(s_i) \bmod v \right)_v \in \mathcal{A}_K$$

and

$$\mathscr{L}_{\mathcal{A}_K}^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}]) = \left((-1)^r \sum_{0 \leq d_1 \leq \cdots \leq d_r < \deg v} \prod_{i=1}^{r} \mathscr{S}_{d_i}^{\bullet}(s_i) \bmod v \right)_v \in \mathcal{A}_K.$$

Both maps take $1 = [\varnothing]$ to 1 and are extended K-linearly to $\mathfrak{H}$.

For every $\mathfrak{s} \in \mathcal{I}$, we have

$$\mathscr{L}_{\mathcal{A}_K}^{\zeta}([\mathfrak{s}]) = \zeta_{\mathcal{A}_K}(\mathfrak{s}), \qquad \mathscr{L}_{\mathcal{A}_K}^{\zeta,\star-\mathrm{inv}}([\mathfrak{s}]) = \zeta_{\mathcal{A}_K}^{\star-\mathrm{inv}}(\mathfrak{s}),$$

and

$$\mathscr{L}_{\mathcal{A}_K}^{\mathrm{Li}}([\mathfrak{s}]) = \mathrm{Li}_{\mathcal{A}_K,\mathfrak{s}}(\mathbf{1}), \qquad \mathscr{L}_{\mathcal{A}_K}^{\mathrm{Li},\star-\mathrm{inv}}([\mathfrak{s}]) = \mathrm{Li}_{\mathcal{A}_K,\mathfrak{s}}^{\star-\mathrm{inv}}(\mathbf{1}).$$

By Carlitz's formula

$$\mathscr{S}_d^{\zeta}(s) = \mathscr{S}_d^{\mathrm{Li}}(s) \qquad (1 \leq s \leq q)$$

(see [Tha04, Theorem 5.9.1]), the element

$$\mathfrak{g}_{\mathcal{A}_K} = \mathscr{L}^{\bullet}_{\mathcal{A}_K}([q]) - L_1\mathscr{L}^{\bullet}_{\mathcal{A}_K}([1, q-1])$$

is independent of the choice of $\bullet \in \{\zeta, \mathrm{Li}\}$.

2.2. **Shuffle and Stuffle Products.** We next recall the two products on $\mathfrak{H}$ corresponding to the ζ- and Li-realizations. In [Tha10], Thakur proved the existence of the *q-shuffle relations*, showing that the product of two ∞-adic multiple zeta values can be expressed as an $\mathbb{F}_p$-linear combination of ∞-adic multiple zeta values of the same total weight. In depth one, Chen later obtained the following explicit formula in [Che15]:

$$\zeta_A(s)\zeta_A(n) = \zeta_A(s,n) + \zeta_A(n,s) + \zeta_A(s+n) + \sum_{\substack{1\le j<s+n\\ q-1\mid j}} \Delta^{[j]}_{s,n}\zeta_A(s+n-j, j)$$

for $s, n \in \mathbb{N}$. Here, we write

$$\Delta^{[j]}_{s,n} = \begin{cases} (-1)^{s-1}\binom{j-1}{s-1} + (-1)^{n-1}\binom{j-1}{n-1}, & q-1 \mid j, \\ 0, & \text{otherwise,} \end{cases}$$

for $s, n, j \in \mathbb{N}$. Based on Chen's formula, Yamamoto observed a recursive construction of the q-shuffle product, which was later formulated and proved by Shi (see [Shi18, Definition 3.1.3 and Theorem 3.1.4]). On the other hand, the corresponding product on the special values of Carlitz multiple polylogarithms $\mathrm{Li}_{\mathfrak{s}}(\mathbf{1})$ is the usual stuffle product [Cha14].

For non-empty $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$, we put

$$\mathfrak{s}^- = (s_2, \dots, s_r), \qquad \varnothing^- = \varnothing.$$

Then we consider the following definition.

Definition 2.3. Let $\bullet \in \{\zeta, \mathrm{Li}\}$. We define two K-bilinear products on $\mathfrak{H}$, the q-shuffle product $*^{\zeta}$ and the stuffle product $*^{\mathrm{Li}}$, by

$$1 *^{\bullet} P = P *^{\bullet} 1 = P$$

and, for non-empty indices $\mathfrak{s} = (s_1, \dots, s_r)$ and $\mathfrak{n} = (n_1, \dots, n_\ell)$, by

$$\begin{aligned} \mathfrak{s} *^{\bullet} \mathfrak{n} = {} & [s_1, \mathfrak{s}^- *^{\bullet} \mathfrak{n}] + [n_1, \mathfrak{s} *^{\bullet} \mathfrak{n}^-] \\ & + [s_1 + n_1, \mathfrak{s}^- *^{\bullet} \mathfrak{n}^-] + D^{\bullet}_{\mathfrak{s}}(\mathfrak{n}). \end{aligned} \tag{1}$$

Here we put $D^{\mathrm{Li}}_{\mathfrak{s}}(\mathfrak{n}) = 0$, and

$$D^{\zeta}_{\mathfrak{s}}(\mathfrak{n}) = \sum_{j=1}^{s_1+n_1-1} \Delta^{[j]}_{s_1,n_1}[s_1+n_1-j, [j] *^{\zeta} (\mathfrak{s}^- *^{\zeta} \mathfrak{n}^-)]. \tag{2}$$

For every non-empty $\mathfrak{s} \in \mathcal{I}$, we set $D^{\bullet}_{\mathfrak{s}}(1) = 0$ and extend $D^{\bullet}_{\mathfrak{s}}(-)$ to $\mathfrak{H}$ K-linearly. For convenience, we write

$$D^{\bullet}_{c}(-) = D^{\bullet}_{(c)}(-), \qquad D_{\mathfrak{s}}(-) = D^{\zeta}_{\mathfrak{s}}(-).$$

The products $*^{\zeta}$ and $*^{\mathrm{Li}}$ on $\mathfrak{H}$ are compatible with the corresponding ∞-adic realizations as follows.

Theorem 2.4 ([Cha14, Shi18]; see also [CCM23, Proposition 2.7]). *Let $\bullet \in \{\zeta, \mathrm{Li}\}$. For any $P, Q \in \mathfrak{H}$, we have*

$$\mathscr{L}^{\bullet}_{\infty}(P *^{\bullet} Q) = \mathscr{L}^{\bullet}_{\infty}(P)\mathscr{L}^{\bullet}_{\infty}(Q).$$

The finite realizations satisfy the same product formulas. More precisely, we have the following theorem.

Theorem 2.5 ([CM17, §3.1] and [Shi18, §4]). *Let $\bullet \in \{\zeta, \mathrm{Li}\}$. For any $P, Q \in \mathfrak{H}$, we have*

$$\mathscr{L}^{\bullet}_{\mathcal{A}_K}(P *^{\bullet} Q) = \mathscr{L}^{\bullet}_{\mathcal{A}_K}(P)\mathscr{L}^{\bullet}_{\mathcal{A}_K}(Q).$$

2.3. **Linear Relations.** We next recall the basis theorem for ∞-adic multiple zeta values that will be used in the finite comparison. In [Tod18, Conjecture 7.1], Todd first conjectured a formula for $\dim_K \mathfrak{Z}_{\infty,w}$ for each weight. Later, Thakur predicted an explicit basis indexed by the following set (see [Tha17, Conjecture 8.2]):

$$\mathcal{I}^{\mathrm{T}}_w = \begin{cases} \{\varnothing\}, & w = 0, \\ \{(s_1, \dots, s_r) \in \mathcal{I} : s_i \leq q \ (1 \leq i < r),\ s_r < q,\ \mathrm{wt}(\mathfrak{s}) = w\}, & w \geq 1. \end{cases}$$

Let

$$\mathcal{I}^{\mathrm{T}} = \bigcup_{w \geq 0} \mathcal{I}^{\mathrm{T}}_w.$$

For $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}$, one has (see [CCM23, (2.7)])

$$\zeta_A(\mathfrak{s}) = \mathscr{L}^{\zeta}_{\infty}([\mathfrak{s}]) = \mathscr{L}^{\mathrm{Li}}_{\infty}([\mathfrak{s}]) = \mathrm{Li}_{\mathfrak{s}}(\mathbf{1}).$$

Ngo Dac proved that these values span all multiple zeta values over K (see [ND21, Theorem A]), and the resulting basis conjecture was later proved independently in [CCM23, Theorem 1.5] and [IKLNP24, Theorem B]. Precisely, we have the following theorem.

Theorem 2.6 ([CCM23, Theorem 1.5], [IKLNP24, Theorem B]). *Let $\bullet \in \{\zeta, \mathrm{Li}\}$. Then for each $w \geq 0$,*

$$\{\mathscr{L}^{\bullet}_{\infty}([\mathfrak{s}]) : \mathfrak{s} \in \mathcal{I}^{\mathrm{T}}_w\}$$

forms a K-basis of $\mathfrak{Z}_{\infty,w}$.

For the comparison argument we also need the explicit generators for the kernels of the two ∞-adic realizations. These relations were developed in the framework of [CCM23]. We recall the formulation used here.

Definition 2.7 ([CCM23, §3]). For $c \in \mathbb{N}$, $\mathfrak{s} \in \mathcal{I}$, and $\bullet \in \{\zeta, \mathrm{Li}\}$, define the K-linear map

$$\alpha^{\bullet}_{c;\mathfrak{s}} : \mathfrak{H} \longrightarrow \mathfrak{H}, \qquad \alpha^{\bullet}_{c;\mathfrak{s}}(P) = [c, \mathfrak{s} *^{\bullet} P].$$

For $m \geq 1$, put

$$\alpha^{\bullet,m}_{c;\mathfrak{s}} = \underbrace{\alpha^{\bullet}_{c;\mathfrak{s}} \circ \cdots \circ \alpha^{\bullet}_{c;\mathfrak{s}}}_{m \text{ times}}, \qquad \alpha^{\bullet,0}_{c;\mathfrak{s}} = \mathrm{id}_{\mathfrak{H}}.$$

We also define the K-bilinear map

$$\boxplus : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathfrak{H}$$

by

$$1 \boxplus P = P \boxplus 1 = 0$$

and

$$\mathfrak{s} \boxplus \mathfrak{n} = [s_1, \dots, s_{r-1}, s_r + n_1, n_2, \dots, n_\ell]$$

for non-empty indices $\mathfrak{s} = (s_1, \dots, s_r)$ and $\mathfrak{n} = (n_1, \dots, n_\ell) \in \mathcal{I}$. Here 0 denotes the zero vector of $\mathfrak{H}$, rather than the empty word $1 = [\varnothing]$.

For non-empty $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$, we let

$$\mathfrak{s}^+ = (s_1, \dots, s_{r-1}), \qquad \varnothing^+ = \varnothing.$$

Also, for $c \in \mathbb{N}$ and $m \geq 1$, we write $\{c\}^m = (c, \dots, c)$ for the index consisting of m copies of c and $\{c\}^0 = \varnothing$.

Definition 2.8 ([Mis26, §2]; see also [CCM23, §3]). Let $\bullet \in \{\zeta, \mathrm{Li}\}$. Put $\varepsilon_\zeta = 1$ and $\varepsilon_{\mathrm{Li}} = 0$. For $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ and $m \geq 1$, define

$$\begin{aligned}\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) = {} & [\mathfrak{s}, \{q\}^m, \mathfrak{n}] + [\mathfrak{s}, \{q\}^m \boxplus \mathfrak{n}] + \varepsilon_\bullet [\mathfrak{s}, \{q\}^{m-1}, D_q(\mathfrak{n})] \\ & - L_1^m [\mathfrak{s}, \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})] - L_1^m [\mathfrak{s}] \boxplus \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}) \\ & - \varepsilon_\bullet L_1^m [\mathfrak{s}^+, D_{s_r}(\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}))],\end{aligned}$$

if $\mathfrak{s} = (s_1, \dots, s_r)$ is non-empty, and

$$\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n}) = [\{q\}^m, \mathfrak{n}] + [\{q\}^m \boxplus \mathfrak{n}] + \varepsilon_\bullet [\{q\}^{m-1}, D_q(\mathfrak{n})] - L_1^m \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}).$$

We also set

$$\mathscr{R}^\bullet = \mathrm{span}_K \{\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) : \mathfrak{s}, \mathfrak{n} \in \mathcal{I},\ m \geq 1\} \subseteq \mathfrak{H}$$

and

$$\mathscr{R}^\bullet_w = \mathrm{span}_K \left\{ \mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) : \begin{array}{c} \mathfrak{s}, \mathfrak{n} \in \mathcal{I},\ m \geq 1, \\ \mathrm{wt}(\mathfrak{s}) + mq + \mathrm{wt}(\mathfrak{n}) = w \end{array} \right\} \subseteq \mathfrak{H}_w.$$

These relations generate precisely the kernels of the corresponding ∞-adic realization maps.

Proposition 2.9 ([Mis26, Proposition 2.3.2]; see also [CCM23, §3]). *For every $\bullet \in \{\zeta, \mathrm{Li}\}$, $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$, and $m \geq 1$, we have*

$$\mathscr{L}^\bullet_\infty(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})) = 0.$$

Moreover, for all $w \geq 0$,

$$\ker(\mathscr{L}^\bullet_\infty|_{\mathfrak{H}_w} : \mathfrak{H}_w \longrightarrow \mathfrak{Z}_\infty) = \mathscr{R}^\bullet_w, \qquad \ker(\mathscr{L}^\bullet_\infty : \mathfrak{H} \longrightarrow \mathfrak{Z}_\infty) = \mathscr{R}^\bullet.$$

3. The Finite Comparison

3.1. **Formalism of Evaluation Maps.** We recall the formalism of evaluation maps developed in [Tsu26b, §3.1] and state only the results needed below. Throughout this subsection, let $\bullet \in \{\zeta, \mathrm{Li}\}$, let B be a commutative unital K-algebra, and let

$$\mathrm{ev}, \mathrm{ev}^{\star-\mathrm{inv}} : \mathfrak{H} \longrightarrow B$$

be K-linear maps. We consider the following conditions:

(C1) $\mathrm{ev}(1) = \mathrm{ev}^{\star-\mathrm{inv}}(1) = 1$.

(C2) For every non-empty index $\mathfrak{s} \in \mathcal{I}$,

$$m_B(\mathrm{ev} \otimes \mathrm{ev}^{\star-\mathrm{inv}})\Delta([\mathfrak{s}]) = 0 = m_B(\mathrm{ev}^{\star-\mathrm{inv}} \otimes \mathrm{ev})\Delta([\mathfrak{s}]),$$

where m_B denotes the multiplication map of B.

(C3) If $\bullet = \zeta$, then $\mathrm{ev}^{\star-\mathrm{inv}}([j]) = 0$ whenever $q-1 \mid j$, while if $\bullet = \mathrm{Li}$, then $\mathrm{ev}^{\star-\mathrm{inv}}([q-1]) = 0$.

(C4) For all $P, Q \in \mathfrak{H}$,
$$\mathrm{ev}(P *^\bullet Q) = \mathrm{ev}(P)\mathrm{ev}(Q).$$

(C5) For all $P, Q \in \mathfrak{H}$,
$$\mathrm{ev}^{\star-\mathrm{inv}}(P *^\bullet Q) = \mathrm{ev}^{\star-\mathrm{inv}}(P)\mathrm{ev}^{\star-\mathrm{inv}}(Q).$$

Define
$$\mathcal{R} = (\mathrm{id} \otimes \mathrm{ev}^{\star-\mathrm{inv}})\Delta : \mathfrak{H} \longrightarrow \mathfrak{H} \otimes_K B,$$
and equip $\mathfrak{H} \otimes_K B$ with the product
$$(P \otimes x) \star^\bullet (Q \otimes y) = (P *^\bullet Q) \otimes xy$$
for $P, Q \in \mathfrak{H}$ and $x, y \in B$.

We will use the following three lemmas.

Lemma 3.1. *For $\bullet = \mathrm{Li}$, assume Conditions C1, C2, and C4. For $\bullet = \zeta$, assume Conditions C1–C4. Then, for all $P, Q \in \mathfrak{H}$,*
$$\mathcal{R}(P *^\bullet Q) = \mathcal{R}(P) \star^\bullet \mathcal{R}(Q).$$
In particular, Condition C5 holds.

Lemma 3.2. *For every $s \in \mathbb{N}$ and $d \geq 0$, let $\mathscr{S}_d(s) \in B$, and let $D \in \mathbb{N} \cup \{\infty\}$. Assume that the sums below are well-defined in B. Define K-linear maps $\mathrm{ev}, \mathrm{ev}^{\star-\mathrm{inv}} : \mathfrak{H} \to B$ by $\mathrm{ev}(1) = \mathrm{ev}^{\star-\mathrm{inv}}(1) = 1$ and, for every non-empty index $\mathfrak{s} = (s_1, \ldots, s_r)$, by*
$$\mathrm{ev}([\mathfrak{s}]) = \sum_{D > d_1 > \cdots > d_r \geq 0} \prod_{k=1}^{r} \mathscr{S}_{d_k}(s_k),$$
$$\mathrm{ev}^{\star-\mathrm{inv}}([\mathfrak{s}]) = (-1)^r \sum_{0 \leq d_1 \leq \cdots \leq d_r < D} \prod_{k=1}^{r} \mathscr{S}_{d_k}(s_k).$$
Then Conditions C1 and C2 hold.

Lemma 3.3. *Assume Conditions C1 and C2. If Condition C4 holds, then*
$$\mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}) \subseteq \mathrm{ev}(\mathfrak{H}).$$
If Condition C5 holds, then
$$\mathrm{ev}(\mathfrak{H}) \subseteq \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}).$$
In particular, if both Conditions C4 and C5 hold, then
$$\mathrm{ev}(\mathfrak{H}) = \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}).$$

We will also use the following consequence of [Tsu26b, §4.1] (see also [Mis26, Definition 1.3.1 and Proposition 2.2.4]).

Lemma 3.4. *Let $\bullet \in \{\zeta, \mathrm{Li}\}$. Then*
$$\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}(\mathfrak{H}) \subseteq \mathscr{L}_\infty^{\bullet}(\mathfrak{H}) = \mathfrak{Z}_\infty.$$
Moreover,
$$\left\{\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}(P) \bmod \zeta_A(q-1)\mathfrak{Z}_\infty : P \in \mathfrak{H}\right\} = \mathfrak{Z}_\infty/\zeta_A(q-1)\mathfrak{Z}_\infty.$$

3.2. **The Finite Comparison Map.** We now turn to the finite setting. For $\bullet \in \{\zeta, \mathrm{Li}\}$, let $\mathfrak{Z}^\bullet_{\mathcal{A}_K} \subseteq \mathcal{A}_K$ be the K-algebra generated by $\mathscr{L}^\bullet_{\mathcal{A}_K}([\mathfrak{s}])$ with $\mathfrak{s} \in \mathcal{I}$. For $w \geq 0$, put

$$\mathfrak{Z}^\bullet_{\mathcal{A}_K,w} = \mathrm{span}_K \left\{ \mathscr{L}^\bullet_{\mathcal{A}_K}([\mathfrak{s}]) : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w \right\},$$

and set $\mathfrak{Z}^\bullet_{\mathcal{A}_K,w} = 0$ for $w < 0$. We also put

$$\mathfrak{Z}^{\bullet,\mathrm{T}}_{\mathcal{A}_K,w} = \mathrm{span}_K \{ \mathscr{L}^\bullet_{\mathcal{A}_K}([\mathfrak{s}]) : \mathfrak{s} \in \mathcal{I}^{\mathrm{T}}_w \}.$$

By Theorem 2.5, we have

$$\mathfrak{Z}^\bullet_{\mathcal{A}_K} = \sum_{w \geq 0} \mathfrak{Z}^\bullet_{\mathcal{A}_K,w}, \qquad \mathfrak{Z}^\bullet_{\mathcal{A}_K,w} \mathfrak{Z}^\bullet_{\mathcal{A}_K,w'} \subseteq \mathfrak{Z}^\bullet_{\mathcal{A}_K,w+w'}.$$

For $D \in \mathbb{Z}$, we define K-linear maps

$$\mathscr{L}^\bullet_D,\ \mathscr{L}^\bullet_{<D} : \mathfrak{H} \longrightarrow K$$

as follows. We set

$$\mathscr{L}^\bullet_D([\varnothing]) = \begin{cases} 1, & D = -1, \\ 0, & D \neq -1, \end{cases} \qquad \mathscr{L}^\bullet_{<D}([\varnothing]) = \begin{cases} 1, & D \geq 0, \\ 0, & D < 0. \end{cases}$$

For a non-empty index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathcal{I}$, put

$$\mathscr{L}^\bullet_D([\mathfrak{s}]) = \sum_{D = d_1 > d_2 > \cdots > d_r \geq 0} \prod_{i=1}^{r} \mathscr{S}^\bullet_{d_i}(s_i)$$

and

$$\mathscr{L}^\bullet_{<D}([\mathfrak{s}]) = \sum_{D > d_1 > \cdots > d_r \geq 0} \prod_{i=1}^{r} \mathscr{S}^\bullet_{d_i}(s_i).$$

Thus, for every non-empty $\mathfrak{s} = (s_1, \mathfrak{s}^-) \in \mathcal{I}$,

$$\mathscr{L}^\bullet_D([\mathfrak{s}]) = \mathscr{S}^\bullet_D(s_1) \mathscr{L}^\bullet_{<D}([\mathfrak{s}^-]) \qquad (D \geq 0),$$

and

$$\mathscr{L}^\bullet_{<D}([\mathfrak{s}]) = \sum_{0 \leq d < D} \mathscr{L}^\bullet_d([\mathfrak{s}]) \qquad (D \in \mathbb{Z}).$$

In particular, for every $P \in \mathfrak{H}$, we have

$$\mathscr{L}^\bullet_{\mathcal{A}_K}(P) = \left(\mathscr{L}^\bullet_{<\deg v}(P) \bmod v \right)_v \in \mathcal{A}_K.$$

We next recall the notion of binary relations and the operators $\mathscr{B}$ and $\mathscr{C}$, introduced by Todd [Tod18, §3], as well as the $\mathscr{B}\mathscr{C}$ operator introduced by Ngo Dac [ND21]. We use the simultaneous ζ- and Li-formulation of these constructions given in [CCM23, Appendix A.2–A.3] as follows:

Let $\bullet \in \{\zeta, \mathrm{Li}\}$ and

$$\mathfrak{H}^+ = \mathrm{span}_K \{ [\mathfrak{s}] : \varnothing \neq \mathfrak{s} \in \mathcal{I} \}.$$

A pair $(P, Q) \in \mathfrak{H}^+ \times \mathfrak{H}^+$ is called a *binary relation* for the $\bullet$-realization if

$$\mathscr{L}^\bullet_d(P) + \mathscr{L}^\bullet_{d+1}(Q) = 0 \qquad (d \in \mathbb{Z}).$$

The fundamental binary relation is

$$R_1 = ([q], -L_1[1, q-1]),$$

corresponding to

$$\mathscr{L}_d^\bullet([q]) - L_1\mathscr{L}_{d+1}^\bullet([1,q-1]) = 0 \qquad (d \in \mathbb{Z})$$

for both $\bullet = \zeta$ and $\bullet = \mathrm{Li}$ (see [Tha09, Theorem 5] and [CCM23, Appendix A.2]).

We also recall the operators on binary relations following [CCM23, Appendix A.3]. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathcal{I}$ be non-empty, and write

$$P = \sum_{\mathfrak{n} \neq \varnothing} a_{\mathfrak{n}}[\mathfrak{n}], \qquad Q = \sum_{\mathfrak{n} \neq \varnothing} b_{\mathfrak{n}}[\mathfrak{n}].$$

Put $D_{c,Q}^\bullet = \sum_{\mathfrak{n} \neq \varnothing} b_{\mathfrak{n}} D_c^\bullet(\mathfrak{n})$. Then we define

$$\mathscr{B}_{\mathfrak{s}}^\bullet(P,Q) = \left([\mathfrak{s},P] + [\mathfrak{s},Q] + [\mathfrak{s}] \boxplus Q + [\mathfrak{s}^+, D_{s_r,Q}^\bullet],\, 0\right),$$

and

$$\mathscr{C}_{\mathfrak{s}}^\bullet(P,Q) = \sum_{\substack{\mathfrak{n}=(n_1,\mathfrak{n}^-)\in\mathcal{I} \\ \mathfrak{n}\neq\varnothing}} \left(a_{\mathfrak{n}}[n_1+s_1, \mathfrak{n}^- *^\bullet \mathfrak{s}^-] + a_{\mathfrak{n}}[n_1, \mathfrak{n}^- *^\bullet \mathfrak{s}] + a_{\mathfrak{n}} D_{\mathfrak{n}}^\bullet(\mathfrak{s}),\, b_{\mathfrak{n}}[n_1, \mathfrak{n}^- *^\bullet \mathfrak{s}]\right).$$

For $m \geq 0$, we further put

$$\mathscr{BC}_q^{\bullet,m}(P,Q) = \left([\{q\}^m, P],\, L_1^m \alpha_{1;(q-1)}^{\bullet,m}(Q)\right).$$

We set $\mathscr{B}_\varnothing^\bullet = \mathscr{C}_\varnothing^\bullet = \mathrm{id}$. Then these operators preserve binary relations by [CCM23, Proposition A.4]. Finally, let

$$\beta : \mathfrak{H}^+ \times \mathfrak{H}^+ \longrightarrow \mathfrak{H}^+, \qquad \beta(P,Q) = P + Q.$$

Then one notices that, for every $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$ and $m \geq 1$,

$$\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}) = \beta\left(\mathscr{B}_{\mathfrak{s}}^\bullet\left(\mathscr{BC}_q^{\bullet,m-1}\left(\mathscr{C}_{\mathfrak{n}}^\bullet(R_1)\right)\right)\right).$$

Lemma 3.5. *Let $\bullet \in \{\zeta, \mathrm{Li}\}$, $\mathfrak{s}, \mathfrak{n} \in \mathcal{I}$, and $m \geq 1$. Then*

$$\mathscr{L}_{\mathcal{A}_K}^\bullet\left(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})\right) = \begin{cases} 0, & \mathfrak{s} \neq \varnothing, \\ \mathfrak{g}_{\mathcal{A}_K} \mathscr{L}_{\mathcal{A}_K}^\bullet\left(L_1^{m-1} \alpha_{1;(q-1)}^{\bullet,m-1}(\mathfrak{n})\right), & \mathfrak{s} = \varnothing. \end{cases}$$

Consequently, if $P \in \ker(\mathscr{L}_\infty^\bullet)$ is homogeneous of weight w, then

$$\mathscr{L}_{\mathcal{A}_K}^\bullet(P) \in \mathfrak{g}_{\mathcal{A}_K} \mathfrak{Z}_{\mathcal{A}_K, w-q}^\bullet.$$

Proof. Since R_1 is a binary relation and the operators $\mathscr{C}_{\mathfrak{n}}^\bullet$, $\mathscr{BC}_q^{\bullet,m-1}$, and $\mathscr{B}_{\mathfrak{s}}^\bullet$ preserve binary relations (see [CCM23, Proposition A.4]), the pair

$$\mathscr{B}_{\mathfrak{s}}^\bullet\left(\mathscr{BC}_q^{\bullet,m-1}\left(\mathscr{C}_{\mathfrak{n}}^\bullet(R_1)\right)\right)$$

is a binary relation. Suppose first that $\mathfrak{s} \neq \varnothing$. By the definition of $\mathscr{B}_{\mathfrak{s}}^\bullet$, the second component of this binary relation is zero. Thus, we obtain

$$\mathscr{B}_{\mathfrak{s}}^\bullet\left(\mathscr{BC}_q^{\bullet,m-1}\left(\mathscr{C}_{\mathfrak{n}}^\bullet(R_1)\right)\right) = (\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n}), 0)$$

and

$$\mathscr{L}_d^\bullet\left(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})\right) = 0 \qquad (d \in \mathbb{Z}).$$

It follows immediately that

$$\mathscr{L}_{\mathcal{A}_K}^\bullet\left(\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})\right) = 0.$$

Now suppose that $\mathfrak{s} = \varnothing$. Since $\mathscr{B}^\bullet_\varnothing = \mathrm{id}$, write

$$(P, Q) = \mathscr{B}\mathscr{C}_q^{\bullet,m-1}\left(\mathscr{C}^\bullet_{\mathfrak{n}}(R_1)\right).$$

By the definitions of $\mathscr{C}^\bullet_{\mathfrak{n}}$ and $\mathscr{B}\mathscr{C}_q^{\bullet,m-1}$, the second component is

$$Q = -L_1^m \alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n}),$$

while $P + Q = \mathscr{A}^\bullet(\varnothing; m; \mathfrak{n})$. Moreover, for every $d \geq 0$, we have

$$\mathscr{L}^\bullet_d(P) + \mathscr{L}^\bullet_{d+1}(Q) = 0.$$

Using $P = \mathscr{A}^\bullet(\varnothing; m; \mathfrak{n}) - Q$, we obtain

$$\mathscr{L}^\bullet_d\left(\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n})\right) = L_1^m\left(\mathscr{L}^\bullet_{d+1} - \mathscr{L}^\bullet_d\right)\left(\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})\right).$$

Notice that for $m \geq 1$, $\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})$ has depth at least two, and hence

$$\mathscr{L}^\bullet_0(\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})) = 0.$$

For $D \geq 0$, summing over $0 \leq d < D$ gives

$$\mathscr{L}^\bullet_{<D}\left(\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n})\right) = L_1^m \mathscr{L}^\bullet_D\left(\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})\right).$$

By the definition of $\alpha^\bullet_{1;(q-1)}$ and the properties of $\mathscr{L}^\bullet_{<D}$ (see [CCM23, Proposition 2.7]), we have

$$L_1^m \mathscr{L}^\bullet_D\left(\alpha^{\bullet,m}_{1;(q-1)}(\mathfrak{n})\right) = \left(L_1 \mathscr{S}^\bullet_D(1)\mathscr{L}^\bullet_{<D}([q-1])\right) L_1^{m-1}\mathscr{L}^\bullet_{<D}\left(\alpha^{\bullet,m-1}_{1;(q-1)}(\mathfrak{n})\right).$$

On the other hand, summing the fundamental binary relation

$$\mathscr{L}^\bullet_d([q]) - L_1\mathscr{L}^\bullet_{d+1}([1, q-1]) = 0$$

over $0 \leq d < D$ yields

$$L_1\mathscr{S}^\bullet_D(1)\mathscr{L}^\bullet_{<D}([q-1]) = \mathscr{L}^\bullet_{<D}([q]) - L_1\mathscr{L}^\bullet_{<D}([1, q-1]).$$

Consequently,

$$\mathscr{L}^\bullet_{<D}\left(\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n})\right) = \left(\mathscr{L}^\bullet_{<D}([q]) - L_1\mathscr{L}^\bullet_{<D}([1, q-1])\right) L_1^{m-1}\mathscr{L}^\bullet_{<D}\left(\alpha^{\bullet,m-1}_{1;(q-1)}(\mathfrak{n})\right).$$

Therefore, it follows from the definitions that

$$\mathscr{L}^\bullet_{\mathcal{A}_K}\left(\mathscr{A}^\bullet(\varnothing; m; \mathfrak{n})\right) = \mathfrak{g}_{\mathcal{A}_K}\mathscr{L}^\bullet_{\mathcal{A}_K}\left(L_1^{m-1}\alpha^{\bullet,m-1}_{1;(q-1)}(\mathfrak{n})\right) \in \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}^\bullet_{\mathcal{A}_K, w-q}$$

for $w = mq + \mathrm{wt}(\mathfrak{n})$.

Finally, let $P \in \ker(\mathscr{L}^\bullet_\infty)$ be homogeneous of weight w. By Proposition 2.9, P is a K-linear combination of homogeneous elements $\mathscr{A}^\bullet(\mathfrak{s}; m; \mathfrak{n})$ of weight w, and hence

$$\mathscr{L}^\bullet_{\mathcal{A}_K}(P) \in \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}^\bullet_{\mathcal{A}_K, w-q},$$

as desired. □

Corollary 3.6. *For every $\bullet \in \{\zeta, \mathrm{Li}\}$ and $w \geq 0$, we have*

$$\mathfrak{Z}^\bullet_{\mathcal{A}_K, w} = \mathfrak{Z}^{\bullet,\mathrm{T}}_{\mathcal{A}_K, w} + \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}^\bullet_{\mathcal{A}_K, w-q}.$$

Moreover, we have

$$\mathfrak{Z}^\zeta_{\mathcal{A}_K, w} = \mathfrak{Z}^{\mathrm{Li}}_{\mathcal{A}_K, w} = \mathfrak{Z}_{\mathcal{A}_K, w}$$

for every $w \geq 0$. In particular,

$$\mathfrak{Z}^{\zeta}_{\mathcal{A}_K} = \mathfrak{Z}^{\mathrm{Li}}_{\mathcal{A}_K} = \mathfrak{Z}_{\mathcal{A}_K}.$$

Proof. Fix $\bullet \in \{\zeta, \mathrm{Li}\}$ and $w \geq 0$. Let $\mathfrak{s} \in \mathcal{I}$ be an index with $\mathrm{wt}(\mathfrak{s}) = w$. By Theorem 2.6, there exists $P_{\mathfrak{s}} \in \mathrm{span}_K\{[\mathfrak{n}] : \mathfrak{n} \in \mathcal{I}^{\mathrm{T}}_w\}$ such that

$$\mathscr{L}^{\bullet}_{\infty}([\mathfrak{s}]) = \mathscr{L}^{\bullet}_{\infty}(P_{\mathfrak{s}}).$$

Thus $[\mathfrak{s}] - P_{\mathfrak{s}} \in \ker(\mathscr{L}^{\bullet}_{\infty})$. By Lemma 3.5,

$$\mathscr{L}^{\bullet}_{\mathcal{A}_K}([\mathfrak{s}]) - \mathscr{L}^{\bullet}_{\mathcal{A}_K}(P_{\mathfrak{s}}) \in \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}^{\bullet}_{\mathcal{A}_K, w-q}.$$

Thus, we obtain

$$\mathfrak{Z}^{\bullet}_{\mathcal{A}_K, w} \subseteq \mathfrak{Z}^{\bullet,\mathrm{T}}_{\mathcal{A}_K, w} + \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}^{\bullet}_{\mathcal{A}_K, w-q}.$$

Conversely, note that

$$\mathfrak{Z}^{\bullet,\mathrm{T}}_{\mathcal{A}_K, w} \subseteq \mathfrak{Z}^{\bullet}_{\mathcal{A}_K, w} \qquad \text{and} \qquad \mathfrak{g}_{\mathcal{A}_K} = \mathscr{L}^{\bullet}_{\mathcal{A}_K}([q]) - L_1\mathscr{L}^{\bullet}_{\mathcal{A}_K}([1, q-1]) \in \mathfrak{Z}^{\bullet}_{\mathcal{A}_K, q}.$$

Hence,

$$\mathfrak{Z}^{\bullet}_{\mathcal{A}_K, w} = \mathfrak{Z}^{\bullet,\mathrm{T}}_{\mathcal{A}_K, w} + \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}^{\bullet}_{\mathcal{A}_K, w-q}.$$

Next, for every $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}$, note that $\mathscr{L}^{\zeta}_{\mathcal{A}_K}([\mathfrak{s}]) = \mathscr{L}^{\mathrm{Li}}_{\mathcal{A}_K}([\mathfrak{s}])$ and therefore

$$\mathfrak{Z}^{\zeta,\mathrm{T}}_{\mathcal{A}_K, w} = \mathfrak{Z}^{\mathrm{Li},\mathrm{T}}_{\mathcal{A}_K, w}.$$

It follows by induction on w that

$$\mathfrak{Z}^{\zeta}_{\mathcal{A}_K, w} = \mathfrak{Z}^{\mathrm{Li}}_{\mathcal{A}_K, w}$$

for every $w \geq 0$, which completes the proof. □

Theorem 3.7. *There exists a well-defined surjective K-algebra homomorphism*

$$\widetilde{\Phi}_{\infty,\mathcal{A}_K} : \mathfrak{Z}_{\infty} \longrightarrow \mathfrak{Z}_{\mathcal{A}_K}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}$$

such that, for every $\bullet \in \{\zeta, \mathrm{Li}\}$ and $\mathfrak{s} \in \mathcal{I}$,

$$\widetilde{\Phi}_{\infty,\mathcal{A}_K}\left(\mathscr{L}^{\bullet}_{\infty}([\mathfrak{s}])\right) = \mathscr{L}^{\bullet}_{\mathcal{A}_K}([\mathfrak{s}]) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

Moreover,

$$\zeta_A(q-1)\mathfrak{Z}_{\infty} \subseteq \ker\widetilde{\Phi}_{\infty,\mathcal{A}_K}.$$

Hence $\widetilde{\Phi}_{\infty,\mathcal{A}_K}$ induces a well-defined surjective K-algebra homomorphism

$$\Phi_{\infty,\mathcal{A}_K} : \mathfrak{Z}_{\infty}/\zeta_A(q-1)\mathfrak{Z}_{\infty} \longrightarrow \mathfrak{Z}_{\mathcal{A}_K}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

Proof. Fix $\bullet \in \{\zeta, \mathrm{Li}\}$. Define

$$\widetilde{\Phi}^{\bullet}_{\infty,\mathcal{A}_K} : \mathfrak{Z}_{\infty} \longrightarrow \mathfrak{Z}_{\mathcal{A}_K}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}$$

by

$$\mathscr{L}^{\bullet}_{\infty}(P) \longmapsto \mathscr{L}^{\bullet}_{\mathcal{A}_K}(P) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

We first show that this map is well-defined. Suppose that $P, Q \in \mathfrak{H}$ satisfy

$$\mathscr{L}^{\bullet}_{\infty}(P) = \mathscr{L}^{\bullet}_{\infty}(Q).$$

Then $P - Q \in \ker(\mathscr{L}^{\bullet}_{\infty})$. By Proposition 2.9 and Lemma 3.5, it follows that

$$\mathscr{L}^{\bullet}_{\mathcal{A}_K}(P) \equiv \mathscr{L}^{\bullet}_{\mathcal{A}_K}(Q) \pmod{\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}},$$

so $\widetilde{\Phi}^{\bullet}_{\infty,\mathcal{A}_K}$ is well-defined. Moreover, Theorems 2.4 and 2.5 show that $\widetilde{\Phi}^{\bullet}_{\infty,\mathcal{A}_K}$ is a K-algebra homomorphism. By Corollary 3.6, $\widetilde{\Phi}^{\bullet}_{\infty,\mathcal{A}_K}$ is surjective.

Finally, for every $\mathfrak{s} \in \mathcal{I}^{\mathrm{T}}$, we have

$$\mathscr{L}^{\zeta}_{\infty}([\mathfrak{s}]) = \mathscr{L}^{\mathrm{Li}}_{\infty}([\mathfrak{s}]), \qquad \mathscr{L}^{\zeta}_{\mathcal{A}_K}([\mathfrak{s}]) = \mathscr{L}^{\mathrm{Li}}_{\mathcal{A}_K}([\mathfrak{s}]).$$

By Theorem 2.6, these elements span $\mathfrak{Z}_{\infty}$ over K. Hence

$$\widetilde{\Phi}^{\zeta}_{\infty,\mathcal{A}_K} = \widetilde{\Phi}^{\mathrm{Li}}_{\infty,\mathcal{A}_K}.$$

We denote this common map by $\widetilde{\Phi}_{\infty,\mathcal{A}_K}$.

Since

$$\widetilde{\Phi}_{\infty,\mathcal{A}_K}(\zeta_A(q-1)) = \zeta_{\mathcal{A}_K}(q-1) = 0$$

by [Shi18, Proposition 4.1.3(1)], we have

$$\zeta_A(q-1)\mathfrak{Z}_{\infty} \subseteq \ker \widetilde{\Phi}_{\infty,\mathcal{A}_K}.$$

Thus $\widetilde{\Phi}_{\infty,\mathcal{A}_K}$ induces the asserted map $\Phi_{\infty,\mathcal{A}_K}$. □

Lemma 3.8. *Let* $\bullet \in \{\zeta, \mathrm{Li}\}$. *Then*

$$\mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\mathcal{A}_K}(\mathfrak{H}) = \mathscr{L}^{\bullet}_{\mathcal{A}_K}(\mathfrak{H}) = \mathfrak{Z}_{\mathcal{A}_K}.$$

Proof. We apply Lemma 3.3 to the case

$$B = \mathcal{A}_K, \qquad \mathrm{ev} = \mathscr{L}^{\bullet}_{\mathcal{A}_K}, \qquad \mathrm{ev}^{\star-\mathrm{inv}} = \mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\mathcal{A}_K}.$$

By Lemma 3.2, Conditions C1 and C2 hold, while Theorem 2.5 gives Condition C4. We next verify Condition C3. If $\bullet = \zeta$ and $q-1 \mid j$, then

$$\mathrm{ev}^{\star-\mathrm{inv}}([j]) = -\zeta_{\mathcal{A}_K}(j) = 0$$

by [Shi18, Proposition 4.1.3(1)] (see also [Tsu26a, Corollary 3.21]). If $\bullet = \mathrm{Li}$, then

$$\mathrm{ev}^{\star-\mathrm{inv}}([q-1]) = -\mathrm{Li}_{\mathcal{A}_K,q-1}(\mathbf{1}) = -\zeta_{\mathcal{A}_K}(q-1) = 0.$$

Hence, Lemma 3.1 gives Condition C5. Therefore, by Lemma 3.3 and Corollary 3.6, we obtain

$$\mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\mathcal{A}_K}(\mathfrak{H}) = \mathscr{L}^{\bullet}_{\mathcal{A}_K}(\mathfrak{H}) = \mathfrak{Z}_{\mathcal{A}_K}.$$

□

Theorem 3.9. *For every* $\bullet \in \{\zeta, \mathrm{Li}\}$ *and* $\mathfrak{s} \in \mathcal{I}$, *we have*

$$\widetilde{\Phi}_{\infty,\mathcal{A}_K}\left(\mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\infty}([\mathfrak{s}])\right) = \mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\mathcal{A}_K}([\mathfrak{s}]) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

Proof. Fix $\bullet \in \{\zeta, \mathrm{Li}\}$. By Theorem 3.7, $\widetilde{\Phi}_{\infty,\mathcal{A}_K}$ factors through the quotient $\mathfrak{Z}_{\infty}/\zeta_A(q-1)\mathfrak{Z}_{\infty}$. Hence it suffices to show that

$$\Phi_{\infty,\mathcal{A}_K}\left(\mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\infty}([\mathfrak{s}]) \bmod \zeta_A(q-1)\mathfrak{Z}_{\infty}\right) = \mathscr{L}^{\bullet,\star-\mathrm{inv}}_{\mathcal{A}_K}([\mathfrak{s}]) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

We now put

$$B = \mathfrak{Z}_{\infty}/\zeta_A(q-1)\mathfrak{Z}_{\infty} \times \mathfrak{Z}_{\mathcal{A}_K}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}.$$

Define K-linear maps $\mathrm{ev}, \mathrm{ev}^{\star-\mathrm{inv}} : \mathfrak{H} \longrightarrow B$ by

$$\mathrm{ev}(P) = \left(\mathscr{L}^{\bullet}_{\infty}(P) \bmod \zeta_A(q-1)\mathfrak{Z}_{\infty}, \mathscr{L}^{\bullet}_{\mathcal{A}_K}(P) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}\right)$$

and

$$\mathrm{ev}^{\star-\mathrm{inv}}(P) = \left(\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}(P) \bmod \zeta_A(q-1)\mathfrak{Z}_\infty, \mathscr{L}_{\mathcal{A}_K}^{\bullet,\star-\mathrm{inv}}(P) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}\right).$$

This is well-defined by Lemmas 3.8 and 3.4. By Lemma 3.2, Conditions C1 and C2 hold componentwise, while Theorems 2.4 and 2.5 give Condition C4.

Moreover, Condition C3 holds for the ∞-adic component by [Tsu26b, §4.1], and for the finite component by [Shi18, Proposition 4.1.3(1)] (see also [Tsu26a, Corollary 3.21]). Hence Lemma 3.1 gives Condition C5. Therefore, by Lemma 3.3,

$$\mathrm{ev}(\mathfrak{H}) = \mathrm{ev}^{\star-\mathrm{inv}}(\mathfrak{H}).$$

In particular, for every $\mathfrak{s} \in \mathcal{I}$, there exists $P \in \mathfrak{H}$ such that

$$\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}]) \equiv \mathscr{L}_\infty^{\bullet}(P) \pmod{\zeta_A(q-1)\mathfrak{Z}_\infty}$$

and

$$\mathscr{L}_{\mathcal{A}_K}^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}]) \equiv \mathscr{L}_{\mathcal{A}_K}^{\bullet}(P) \pmod{\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}}.$$

Thus, by Theorem 3.7,

$$\begin{aligned}
&\Phi_{\infty,\mathcal{A}_K}\left(\mathscr{L}_\infty^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}]) \bmod \zeta_A(q-1)\mathfrak{Z}_\infty\right)\\
&\quad= \Phi_{\infty,\mathcal{A}_K}\left(\mathscr{L}_\infty^{\bullet}(P) \bmod \zeta_A(q-1)\mathfrak{Z}_\infty\right)\\
&\quad= \mathscr{L}_{\mathcal{A}_K}^{\bullet}(P) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}\\
&\quad= \mathscr{L}_{\mathcal{A}_K}^{\bullet,\star-\mathrm{inv}}([\mathfrak{s}]) \bmod \mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K},
\end{aligned}$$

as desired. □

4. Dimension Formulas and a Comparison Conjecture

4.1. **Dimension Formulas.** We now compare the dimensions of the spaces of ∞-adic, v-adic, and finite multiple zeta values.

We fix a finite place v of K and consider the notion of *v-adic multiple zeta values*. We refer to [CM19, CM21, CCM22] for details. We write $\mathfrak{Z}_v$ for the K-algebra generated by all v-adic multiple zeta values $\zeta_A(\mathfrak{s})_v$ and

$$\mathfrak{Z}_{v,w} = \mathrm{span}_K\{\zeta_A(\mathfrak{s})_v : \mathfrak{s} \in \mathcal{I},\ \mathrm{wt}(\mathfrak{s}) = w\}$$

for all $w \geq 0$.

We define the following three power series:

$$\begin{aligned}
H_\infty(X) &= \frac{1-X^q}{1-X-X^2-\cdots-X^q} = \sum_{w\geq 0} D_{\infty,w}X^w,\\
H_v(X) &= \frac{(1-X^{q-1})(1-X^q)}{1-X-X^2-\cdots-X^q} = \sum_{w\geq 0} D_{v,w}X^w,\\
H_{\mathcal{A}_K}(X) &= \frac{1-X^{q-1}}{1-X-X^2-\cdots-X^q} = \sum_{w\geq 0} D_{\mathcal{A}_K,w}X^w.
\end{aligned}$$

One checks that

$$D_{\infty,w} = \begin{cases} 1, & w = 0, \\ 2^{w-1}, & 1 \leq w < q, \\ 2^{w-1} - 1, & w = q, \\ \sum_{i=1}^{q} D_{\infty,w-i}, & w > q, \end{cases}$$

$$D_{v,w} = \begin{cases} 1, & w = 0, \\ 2^{w-1}, & 1 \leq w < q-1, \\ 2^{q-2} - 1, & w = q-1, \\ 2^{q-1} - 2, & w = q, \\ \sum_{i=1}^{q} D_{v,w-i} + 1, & w = 2q-1, \\ \sum_{i=1}^{q} D_{v,w-i}, & w > q,\ w \neq 2q-1, \end{cases}$$

and

$$D_{\mathcal{A}_K,w} = \begin{cases} 1, & w = 0, \\ 2^{w-1}, & 1 \leq w < q-1, \\ 2^{w-1} - 1, & w = q-1, q, \\ \sum_{i=1}^{q} D_{\mathcal{A}_K,w-i}, & w > q. \end{cases}$$

These three sequences are closely related to the dimensions of the corresponding spaces of multiple zeta values.

For ∞-adic multiple zeta values, Todd conjectured [Tod18, Conjecture 7.1] that

$$\dim_K \mathfrak{Z}_{\infty,w} = D_{\infty,w} = \#\mathcal{I}_w^{\mathrm{T}} \qquad (w \geq 0),$$

which now follows from Theorem 2.6.

For v-adic multiple zeta values, Chang, Chen, and Mishiba [CCM22, Theorem 1.2.3] (see also [CM19, Theorem 6.4.1]) constructed a surjective K-algebra homomorphism

$$\Phi_{\infty,v} : \mathfrak{Z}_\infty / \zeta_A(q-1)\mathfrak{Z}_\infty \longrightarrow \mathfrak{Z}_v$$

satisfying

$$\Phi_{\infty,v}\left(\zeta_A(\mathfrak{s}) \bmod \zeta_A(q-1)\mathfrak{Z}_\infty\right) = \zeta_A(\mathfrak{s})_v$$

for every $\mathfrak{s} \in \mathcal{I}$. They further conjectured that this map is an isomorphism.

Conjecture 4.1 ([CCM22, Conjecture 5.4.1])**.** *The map*

$$\Phi_{\infty,v} : \mathfrak{Z}_\infty / \zeta_A(q-1)\mathfrak{Z}_\infty \xrightarrow{\sim} \mathfrak{Z}_v$$

is an isomorphism of K-algebras.

In particular, this conjectural isomorphism predicts that

$$\dim_K \mathfrak{Z}_{v,w} = D_{v,w} \qquad (w \geq 0),$$

while the corresponding upper bound

$$\dim_K \mathfrak{Z}_{v,w} \leq D_{v,w} \qquad (w \geq 0)$$

was established in [CCM23, Corollary 1.8].

For finite multiple zeta values over K, Shi [Shi18] proposed, based on explicit computations of relations among truncated multiple zeta values, the following dimension conjecture.

Conjecture 4.2 ([Shi18, Conjecture 4.6.1])**.** *For every $w \geq 0$, one has*

$$\dim_K \mathfrak{Z}_{\mathcal{A}_K,w} = D_{\mathcal{A}_K,w}.$$

Using the comparison map constructed in Theorem 3.7, we obtain the corresponding upper bound.

Theorem 4.3. *For every $w \geq 0$, we have*

$$\dim_K \mathfrak{Z}_{\mathcal{A}_K,w} \leq D_{\mathcal{A}_K,w}.$$

Proof. Put $\mathfrak{Z}_{\infty,w} = \mathfrak{Z}_{\mathcal{A}_K,w} = 0$ for $w < 0$. By the construction in the proof of Theorem 3.7, together with Lemma 3.5, there exists a surjective K-linear map

$$\mathfrak{Z}_{\infty,w}/\zeta_A(q-1)\mathfrak{Z}_{\infty,w-(q-1)} \longrightarrow \mathfrak{Z}_{\mathcal{A}_K,w}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K,w-q}.$$

Since $\zeta_A(q-1) \neq 0$, we have

$$\dim_K \mathfrak{Z}_{\infty,w}/\zeta_A(q-1)\mathfrak{Z}_{\infty,w-(q-1)} = D_{\infty,w} - D_{\infty,w-(q-1)} = D_{v,w}.$$

Thus, we obtain

$$\begin{aligned}\dim_K \mathfrak{Z}_{\mathcal{A}_K,w} &= \dim_K \mathfrak{Z}_{\mathcal{A}_K,w}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K,w-q} + \dim_K \left(\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K,w-q}\right) \\ &\leq D_{v,w} + \dim_K \mathfrak{Z}_{\mathcal{A}_K,w-q}.\end{aligned}$$

Since $H_v(X) = (1 - X^q)H_{\mathcal{A}_K}(X)$, we have

$$D_{v,w} = D_{\mathcal{A}_K,w} - D_{\mathcal{A}_K,w-q},$$

where $D_{\mathcal{A}_K,w-q} = 0$ if $w < q$. We prove the result by induction on w. For $w < q$, the assertion is immediate. For $w \geq q$, the induction hypothesis gives

$$\dim_K \mathfrak{Z}_{\mathcal{A}_K,w} \leq D_{\mathcal{A}_K,w} - D_{\mathcal{A}_K,w-q} + D_{\mathcal{A}_K,w-q} = D_{\mathcal{A}_K,w}.$$

□

Remark 4.4. In fact, by the degree-wise form of Thakur's fundamental relation (see [Tha09]), we have

$$\mathfrak{g}_{\mathcal{A}_K} = (S_{\deg v-1}(q) \bmod v)_v = \left(\frac{1}{L^q_{\deg v-1}} \bmod v\right)_v.$$

Note that since $v \nmid L_{\deg v-1}$, $\mathfrak{g}_{\mathcal{A}_K} \in \mathcal{A}_K^\times$. In particular, multiplication by $\mathfrak{g}_{\mathcal{A}_K}$ is injective on $\mathfrak{Z}_{\mathcal{A}_K}$, and hence

$$\dim_K \left(\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K,w-q}\right) = \dim_K \mathfrak{Z}_{\mathcal{A}_K,w-q}$$

for every $w \geq 0$.

4.2. **A Unified Comparison Conjecture.** Combining Conjectures 1.2 and 4.1, we are led to the following unified comparison conjecture.

Conjecture 4.5. *The comparison maps*

$$\mathfrak{Z}_v \xleftarrow[\sim]{\Phi_{\infty,v}} \mathfrak{Z}_\infty/\zeta_A(q-1)\mathfrak{Z}_\infty \xrightarrow[\sim]{\Phi_{\infty,\mathcal{A}_K}} \mathfrak{Z}_{\mathcal{A}_K}/\mathfrak{g}_{\mathcal{A}_K}\mathfrak{Z}_{\mathcal{A}_K}$$

are K-algebra isomorphisms.

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