A Direct Sum Theorem for Multiple Eisenstein Series in Positive Characteristic

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Drinfeld Symmetric Spaces and A-Lattices

$$A = \mathbb{F}_q[\theta] \subset K = \mathbb{F}_q(\theta) \subset K_\infty = \mathbb{F}_q((\theta^{-1})) \subset \mathbb{C}_\infty = \widehat{\overline{K_\infty}}.$$

Notes: This is the positive characteristic analogue of $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Drinfeld symmetric space of rank r

$$\Omega^r = \{\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}_\infty^r : \ z_1, \dots, z_r \text{ are } K_\infty\text{-linearly independent}, \ z_r = 1\}.$$

Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}_\infty$. For $\mathbf{z} \in \Omega^r$, put

$$\Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_r \subset \mathbb{C}_\infty, \qquad \langle \mathbf{f}, \mathbf{z} \rangle = \sum_{j=1}^{r} f_j z_j \quad (\mathbf{f} \in A^r).$$

Notes: $\Omega^2 = \mathbb{C}_\infty \setminus K_\infty$ and $\Lambda_{(z,1)} = Az + A$ play the roles of $\mathbb{H}$ and $\mathbb{Z}\tau + \mathbb{Z}$.

Partial Order on Lattices

Set $A_+ = \{f \in A : f \text{ monic}\}$. For $0 \neq \mathbf{f} = (f_1, \dots, f_r) \in A^r$, let

$$\ell(\mathbf{f}) = \min\{i : f_i \neq 0\}.$$

The order on A^r

$$\mathbf{f} \succ 0 \iff f_{\ell(\mathbf{f})} \in A_+,$$

$$\mathbf{f} \succ \mathbf{g} \quad (\mathbf{f}, \mathbf{g} \succ 0) \iff \begin{cases} \ell(\mathbf{f}) < \ell(\mathbf{g}), \text{ or} \\ \ell(\mathbf{f}) = \ell(\mathbf{g}) \text{ and } \deg f_{\ell(\mathbf{f})} > \deg g_{\ell(\mathbf{g})}. \end{cases}$$

Notes: This is the positive characteristic analogue of the lexicographic order on $\mathbb{Z}\tau + \mathbb{Z}$ used to define classical MES.

Multiple Eisenstein Series (MES)

Let $\mathfrak{s} = (s_1, \dots, s_m) \in \mathbb{N}^m$ with $\mathrm{wt}(\mathfrak{s}) = s_1 + \cdots + s_m$.

Definition ([Che17] for $r = 2, m \leq 2$; [CCHT25] for arbitrary $r, m \geq 1$)

For $\mathbf{z} \in \Omega^r$, define

$$E_r(\mathfrak{s}; \mathbf{z}) = \sum_{\substack{\mathbf{f}_1, \dots, \mathbf{f}_m \in A^r \\ \mathbf{f}_1 \succ \cdots \succ \mathbf{f}_m \succ 0}} \frac{1}{\langle \mathbf{f}_1, \mathbf{z} \rangle^{s_1} \cdots \langle \mathbf{f}_m, \mathbf{z} \rangle^{s_m}} \in \mathcal{O}(\Omega^r).$$

We set $\mathrm{wt}(\varnothing) = 0$ and $E_r(\varnothing, \mathbf{z}) = 1$ by convention.

Rank one: $\Omega^1 = \{1\}$ gives Thakur's multiple zeta values (MZVs) [Tha04]:

$$E_1(\mathfrak{s}) = \zeta_A(\mathfrak{s}) = \sum_{\substack{f_1, \dots, f_m \in A_+ \\ \deg f_1 > \cdots > \deg f_m}} \frac{1}{f_1^{s_1} \cdots f_m^{s_m}} \in \mathbb{C}_\infty.$$

The Algebra of MES

For $K \subseteq L \subseteq \mathbb{C}_\infty$ and $w \geq 0$ let

$$\mathcal{Z}_{L,w}^{(r)} = \operatorname{span}_L\{E_r(\mathfrak{s};\mathbf{z}) : \operatorname{wt}(\mathfrak{s}) = w\}, \qquad \mathcal{Z}_L^{(r)} = \sum_{w\geq 0} \mathcal{Z}_{L,w}^{(r)}.$$

q-shuffle product ([Tha10] for $r = 1$; [CCHT25] for $r \geq 2$)

$$\mathcal{Z}_{L,w_1}^{(r)} \cdot \mathcal{Z}_{L,w_2}^{(r)} \subseteq \mathcal{Z}_{L,w_1+w_2}^{(r)}.$$

In particular, $\mathcal{Z}_L^{(r)}$ is an L-algebra.

Is this algebra graded by weight?

$$\mathcal{Z}_K^{(r)} \stackrel{?}{=} \bigoplus_{w\geq 0} \mathcal{Z}_{K,w}^{(r)}$$

Main Theorem

Theorem ([CCHT26])

For every rank $r \geq 1$, we have

1. **Direct sum:** There are no nontrivial $\overline{K}$-linear relations among MES of distinct weights:
$$\mathcal{Z}^{(r)}_{\overline{K}} = \bigoplus_{w \geq 0} \mathcal{Z}^{(r)}_{\overline{K},w}.$$
2. **Descent of relations:** A K-linearly independent family of MES remains $\overline{K}$-linearly independent:
$$\mathcal{Z}^{(r)}_{K} \otimes_K \overline{K} \xrightarrow{\sim} \mathcal{Z}^{(r)}_{\overline{K}}.$$

Base case: Chang [Cha14]. Classical MES counterpart: Bachmann–Kanno [BK26].

The Expansion at Infinity

Write $\mathbf{z} = (z_1, \mathbf{z}') \in \Omega^{r+1}$, where $\mathbf{z}' \in \Omega^r$, and set $\Lambda = \Lambda_{\mathbf{z}'} = Az_2 + \cdots + Az_{r+1}$.

The local parameter

$$t = t_\Lambda(z_1) := \sum_{\lambda \in \Lambda} \frac{1}{z_1 + \lambda} = \frac{1}{\exp_\Lambda(z_1)}, \qquad \exp_\Lambda(X) = X \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{X}{\lambda}\right).$$

For $r \geq 1$, MES have a unique t-expansion near this cusp:

$$E_{r+1}(\mathfrak{s}; \mathbf{z}) = \sum_{N \geq 0} \beta_{\mathfrak{s},N}(\mathbf{z}')\, t^N, \qquad \beta_{\mathfrak{s},0} = E_r(\mathfrak{s}; \mathbf{z}').$$

Notes: This is the positive characteristic analogue of the classical Fourier expansion, with $t_\Lambda(z_1)$ playing the role of $e^{2\pi i \tau}$. The coefficients belong to $\mathcal{O}(\Omega^r)$.

The Coefficients of the t-Expansion

Let $\Delta_r \in \mathcal{O}(\Omega^r)$ be the "Drinfeld discriminant form" of rank r. Then Δ_r is a Drinfeld cusp form of weight $q^r - 1$ for $r \geq 2$ and is nowhere vanishing on Ω^r for $r \geq 1$.

Notes: Δ_r is the positive characteristic analogue of the classical discriminant form $\Delta(\tau)$.

Key theorem ([CCHT26, Theorem 3.6])

For $\mathrm{wt}(\mathfrak{s}) = w$ and every $N \geq 0$, there is an integer $M \geq 0$ such that

$$\beta_{\mathfrak{s},N}(\mathbf{z}') = \frac{F_{\mathfrak{s},N}(\mathbf{z}')}{\Delta_r(\mathbf{z}')^M}, \qquad F_{\mathfrak{s},N} \in \mathcal{Z}^{(r)}_{K,\, w-N+M(q^r-1)}.$$

In other words, after clearing a power of Δ_r, each coefficient is a K-linear combination of rank r MES of a precise weight.

Proof Idea: Direct Sum

We use induction on the rank r.

Base case: $r = 1$ follows from Chang's theorem [Cha14].

Direct sum: Suppose there is a relation among distinct weights:

$$F_1 + \cdots + F_m = 0, \quad F_j \in \mathcal{Z}^{(r+1)}_{\overline{K}, w_j}, \quad w_1 < \cdots < w_m.$$

Write $F_j(\mathbf{z}) = \sum_{N \geq 0} c_{j,N}(\mathbf{z}')t^N$. Uniqueness of the t-expansion gives $\sum_j c_{j,N} = 0$ for each N. By the key theorem, for each N, choose $M \geq 0$ such that

$$\Delta_r^M c_{j,N} \in \mathcal{Z}^{(r)}_{\overline{K}, w_j - N + M(q^r - 1)} \qquad (1 \leq j \leq m).$$

These weights are still distinct. By the induction hypothesis, $c_{j,N} = 0$ for all j, N, hence $F_j = 0$.

Proof Idea: Descent of Relations

Descent of relations: Take K-linearly independent $F_1, \dots, F_m \in \mathcal{Z}_K^{(r+1)}$. By the key theorem, write

$$F_j(\mathbf{z}) = \sum_{N \geq 0} \Delta_r^{-M(N)} d_{j,N}(\mathbf{z}') t^N, \qquad d_{j,N} \in \mathcal{Z}_K^{(r)}.$$

For each $N \geq 0$, define the K-linear map

$$\psi_N : K^{\oplus m} \longrightarrow \mathcal{Z}_K^{(r)}, \qquad (c_1, \dots, c_m) \longmapsto \sum_{j=1}^{m} c_j d_{j,N}.$$

By the uniqueness of the t-expansion and independence of the F_i,

$$\bigcap_{N \geq 0} \ker \psi_N = \{0\}.$$

Proof Idea: Descent of Relations (continued)

Since $K^{\oplus m}$ is finite-dimensional, there exists $N_0 \geq 0$ such that $\bigcap_{N=0}^{N_0} \ker \psi_N = \{0\}$. Define the K-linear map

$$\Psi : K^{\oplus m} \longrightarrow \left(\mathcal{Z}_K^{(r)}\right)^{\oplus(N_0+1)}, \qquad \mathbf{c} \longmapsto (\psi_0(\mathbf{c}), \ldots, \psi_{N_0}(\mathbf{c})).$$

By the choice of N_0, the map Ψ is injective. By the induction hypothesis and the flatness of $\overline{K}$ over K, base change gives an injection

$$\Psi_{\overline{K}} : \overline{K}^{\oplus m} \hookrightarrow \left(\mathcal{Z}_{\overline{K}}^{(r)}\right)^{\oplus(N_0+1)} \cong \left(\mathcal{Z}_K^{(r)}\right)^{\oplus(N_0+1)} \otimes_K \overline{K}.$$

If $\sum_{j=1}^m c_j F_j = 0$ for $c_j \in \overline{K}$, comparing the coefficients in the t-expansion, we have

$$(c_1, \ldots, c_m) \in \ker \Psi_{\overline{K}} \implies c_1 = \cdots = c_m = 0.$$

Thus, $F_1, \ldots, F_m$ remain $\overline{K}$-linearly independent.

The Main Theorem over $\mathbb{C}_\infty$

In fact, using the ideas of Bachmann and Kanno [BK26], we can prove the main theorem over $\mathbb{C}_\infty$ for $r = 2$. Then, applying the above induction argument with base case $r = 2$, we obtain the following result:

Theorem ([CCHT26, Theorem A.1])

For every rank $r \geq 2$, we have

1. **Direct sum:**
$$\mathcal{Z}^{(r)}_{\mathbb{C}_\infty} = \bigoplus_{w \geq 0} \mathcal{Z}^{(r)}_{\mathbb{C}_\infty, w}.$$
2. **Descent of relations:**
$$\mathcal{Z}^{(r)}_K \otimes_K \mathbb{C}_\infty \xrightarrow{\sim} \mathcal{Z}^{(r)}_{\mathbb{C}_\infty}.$$

Thank you!

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